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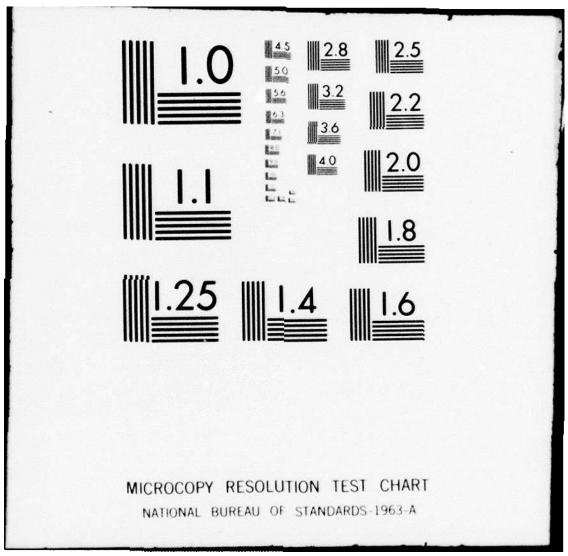
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DYNAMIC THEORY OF PRODUCTION CORRESPONDENCES: PART I

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by

RONALD W. SHEPHARD

and

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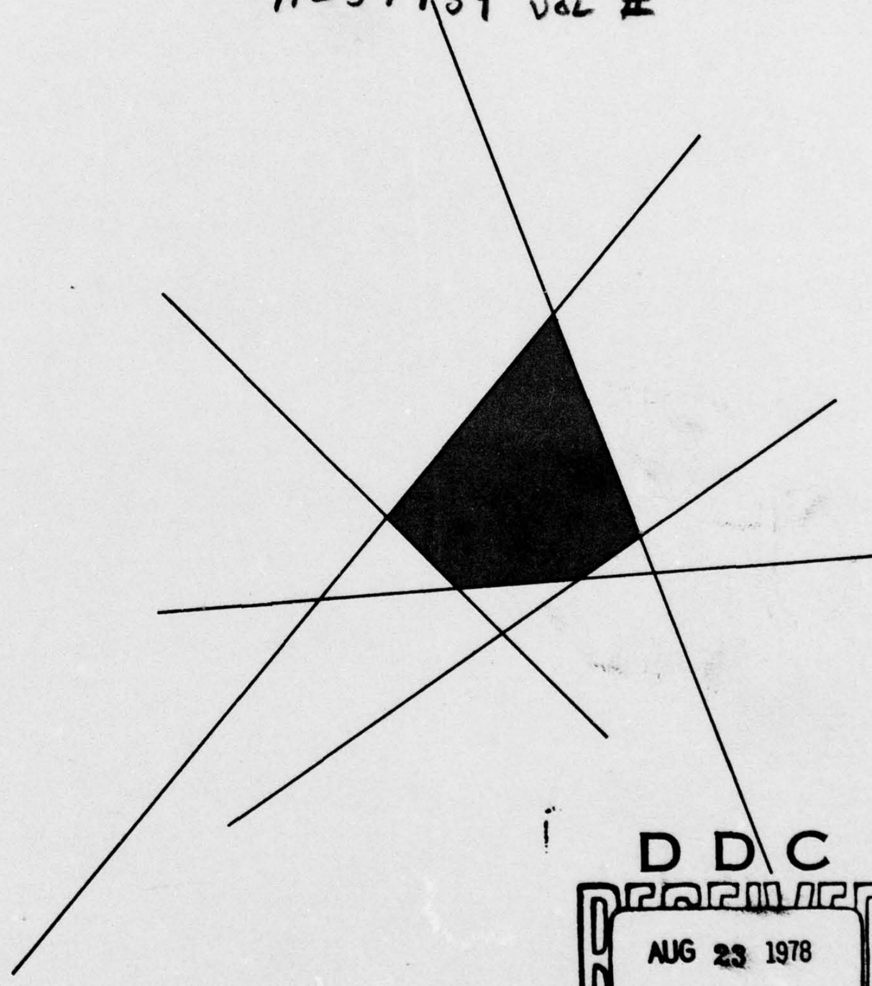

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DYNAMIC THEORY OF PRODUCTION CORRESPONDENCES†

PART I

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ABSTRACT

The first two chapters of a monograph on a Dynamic Theory of Production Correspondences are presented. The indivisible elements are time histories of inputs and outputs, related by mappings between products of function spaces. Axioms are stated for the mappings (correspondences) and compatibility and independence of the axioms are shown. An iff condition is shown for the existence of a dynamic neoclassical production function. Under certain conditions it is shown that steady state production correspondences exist, and definition in terms of the dynamic production correspondence is given.

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CHAPTER 1

THE FUNCTION SPACES OF INPUTS AND OUTPUTS

For the i^{th} exogenous input rate history, consider a real scalar function f_i defined on the interval $R_+ = [0,+\infty)$,

$$f_i : f_i(t) \ , \ t \in [0,+\infty) \ .$$

The function f_i is taken to be an element of the function space $L_\infty\left(R_+, \sum_i, \mu_i\right)$, where $\left(R_+, \sum_i, \mu_i\right)$ is a positive sigma-finite measure space, i.e., f_i is bounded except on a subset of R_+ of μ_i-measure zero, and f_i. is μ_i-measurable with norm

$$||f_i|| = \mu_i\text{-Essential} \sup_t |f_i(t)| \tag{1-1}$$

defined by

$$\mu_i\text{-Essential} \sup_t |f_i(t)| = \text{Inf} \ \{\alpha \in R_+ : \mu_i(S(\alpha)) = 0\}$$

where

$$S(\alpha) = \{t \in R_+ : |f_i(t)| > \alpha \ , \ \alpha \in R_+\} \ .$$

The elements of $L_\infty\left(R_+, \sum_i, \mu_i\right)$ are not actually functions, but equivalence classes of functions. For the purposes of this theory any function of an equivalence class is not distinguishable from another of that class, because the difference between them is a μ_i-null function, i.e., when summed over any interval the result is zero, and isolated instantaneous variations occurring on subsets of R_+ of

μ_i-measure zero have no significance for output.

A partial ordering is defined for any two functions f_i and g_i of $L_\infty\left(R_+, \sum_i, \mu_i\right)$ by:

(1-2) $f_i \geqq g_i$ iff $f_i(t) \geqq g_i(t)$ $\forall\, t \in R_+$, except on a subset of R_+ of μ_i-measure zero - Essentially Equal to or Greater.

(1-3) $f_i \gneq g_i$ iff $f_i \geqq g_i$ with $f_i(t) > g_i(t)$ on a subset of R_+ of positive μ_i-measure - Essentially Semi-Greater.

(1-4) $f_i > g_i$ iff $f_i(t) > g_i(t)$ $\forall\, t \in R_+$ except on a subset of R_+ of μ_i-measure zero - Essentially Greater.

For n factors of production (exogenous inputs) consider vectors

$$f = (f_1, f_2, \ldots, f_n)\ ,\ n \geqq 1\ , \tag{1-5}$$

with f an element of the product space

$$\left(L_\infty\left(R_+, \sum, \mu\right)\right)^n = L_\infty\left(R_+, \sum_1, \mu_1\right) \times \cdots \times L_\infty\left(R_+, \sum_n, \mu_n\right) .$$

The norm for $\left(L_\infty\left(R_+, \sum, \mu\right)\right)^n$ may be taken as

$$||f|| = \max_i \{||f_i|| : i \in \{1,2, \ldots, n\}\} . \tag{1-6}$$

Under the operations

$$f + g = (f_1 + g_1, f_2 + g_2, \ldots, f_n + g_n)$$

$$\alpha f = (\alpha f_1, \alpha f_2, \ldots, \alpha f_n)$$

$$f_i + g_i = f_i(t) + g_i(t) \ , \ t \in R_+ \ , \ i \in \{1,2, \ldots, n\}$$

$$\alpha f_i = \alpha f_i(t) \ , \ t \in R_+ \ i \in \{1,2, \ldots, n\} \ ,$$

$\left(L_\infty\left(R_+, \sum, \mu\right)\right)^n$ is a complete normed vector space, i.e., a Banach space.

A partial ordering is defined for $\left(L_\infty\left(R_+, \sum, \mu\right)\right)^n$ by

(1-7) $$f \geqq g \text{ iff } f_i \geqq g_i \ \forall i \in \{1,2, \ldots, n\} \ .$$

(1-8) $$f \geq g \text{ iff } f \geqq g \text{ but } f_i \geq g_i \text{ for some } i \in \{1,2, \ldots, n\} \ .$$

(1-9) $$f > g \text{ iff } f_i \geq g_i \ \forall i \in \{1,2, \ldots, n\} \ .$$

(1-10) $$f >> g \text{ iff } f > g \text{ and } f_i > g_i \text{ for some } i \in \{1,2, \ldots, n\} \ .$$

(1-11) $$f >>> g \text{ iff } f_i > g_i \ \forall i \in \{1,2, \ldots, n\} \ .$$

With this framework, a vector $x = (x_1, x_2, \ldots, x_n)$ of input time histories

(1-12) $$x_i : x_i(t) \ , \ t \in R_+ \ , \ i \in \{1,2, \ldots, n\}$$

is an element of the subset (domain) of nonnegative functions of $\left(L_\infty\left(R_+, \sum, \mu\right)\right)^n$ defined by:

(1-13) $$\left(L_\infty\left(R_+, \sum, \mu\right)\right)^n_+ := \left\{f \in \left(L_\infty\left(R_+, \sum, \mu\right)\right)^n : f \geqq 0\right\} \ .$$

By use of the metric $\rho(f,g) = ||f - g||$, the vector space $\left(L_\infty \ R_+, \sum, \mu\right)\Big)^n$ is a metric space with norm topology, consisting of all open spheres generated by $\rho(f,g) < \alpha$, $\alpha \in R_+$.

In an entirely similar way a vector $u = (u_1, u_2, \ldots, u_m)$ of output time histories

$$u_i : u_i(t) \ , \ t \in R_+ \ , \ i \in \{1,2, \ldots, m\} \tag{1-14}$$

for m net outputs may be expressed as an element of the nonnegative domain $(L_\infty)^m_+$ of a Banach space $(L_\infty)^m : = L_\infty(R_+,\sigma_1,\nu_1) \times L_\infty(R_+,\sigma_2,\nu_2) \times \cdots \times L_\infty(R_+,\sigma_m,\nu_m)$ where (R_+,σ_i,ν_i) is a positive sigma-finite measure space. A function u_i is an equivalence class of $L_\infty(R_+,\sigma_i,\nu_i)$, with no account taken of differences on subsets of R_+ of ν_i-measure zero. Such differences contribute nothing to accumulated output, and inversely they are not regarded as implying any response required for input histories. The norm of an output history u_i is

$$||u_i|| = \nu_i\text{-Essential} \sup_t |u_i(t)| \tag{1-15}$$

and

$$||u|| = \max_i \{||u_i|| : i \in \{1,2, \ldots, m\}\} \ . \tag{1-16}$$

In production, an output history u_i might be realized as amounts occurring at discrete points of time, with infinite rate of output so to speak when a Lebesgue measure is used for subsets of the interval $t \in [0,+\infty)$. In this case the measure ν_i taken for the family σ_i of subsets of $[0,+\infty)$ is a counting measure, i.e., $\nu_i(A)$, $A \subset [0,+\infty)$ is the number of points of A with positive value of u_i . By this artifice discretely appearing outputs are thought of as occurring at constant rate (equal to the amount of the output) over one unit of time,

which often is all how one would make computations in a practical case for systems of the type considered here. The same kind of treatment will be used for discrete inputs.

Output histories are partially ordered by

$$u_i \geqq v_i \text{ : Essentially Equal to or Greater}$$

$$u_i \geq v_i \text{ : Essentially Semi-Greater}$$

$$u_i > v_i \text{ : Essentially Greater}$$

and vectors of output histories are partially ordered by

$$u \geqq v \iff u_i \geqq v_i \quad \forall i \in \{1,2, \ldots, m\}$$

$$u \geq v \iff u \geqq v \text{ but } u_i \geq v_i \text{ for some } i \in \{1,2, \ldots, m\}$$

$$u > v \iff u_i \geq v_i \quad \forall i \in \{1,2, \ldots, m\}$$

$$u >> v \iff u > v \text{ and } u_i > v_i \text{ for some } i \in \{1,2, \ldots, m\}$$

$$u >>> v \iff u_i > v_i \quad \forall i \in \{1,2, \ldots, m\} .$$

The primitive elements of the theory to follow are time histories of input and output rates, ordered as described above with essential supremum norms and the maximum product norm. Production is regarded as a process of applying vectors of input rate histories to obtain vectors of output rate time histories. In this way the dynamic structure of production will be modeled by a Dynamic Correspondence, i.e., a mapping of vectors of input rate functions (histories) to subsets of vectors of

output rate functions (histories). Corresponding to a vector of input rate functions, there need not be only one ray-distinct vector of output rate functions, i.e., all output rate vectors are not necessarily of the form (λu^o) , $\lambda \in [0,+\infty)$.

The same total amount of an input, say $\hat{x}_i^o$, may be applied over an interval $[0,T)$, $T \in (0,+\infty]$, in a very large number of ways (Time Distributions), given by

$$\left\{ x_i \in L_\infty\left(R_+, \sum_i, u_i\right)_+ : \int_0^T x_i(t)d\mu_i(t) = \hat{x}_i^o \right\},$$

and these time distributions will strongly affect the output histories attainable. Thus, in a dynamic sense, one cannot rigorously model the phenomena of production except by use of a Dynamic Correspondence. In such treatment, the correspondence will be defined as a mapping between the nonnegative domains of normed linear topological spaces of essentially bounded equivalence classes relative to sigma-finite. measure spaces $\left(R_+, \sum_i, \mu_i\right)$ for inputs and (R_+, σ_i, ν_i) for outputs.

In this context a steady state model is one where only the constant rate functions

$$\bar{x}_i : x_i(t) = \bar{x}_i \ , \ t \in [0,+\infty)$$

$$\bar{u}_i : u_i(t) = \bar{u}_i \ , \ t \in [0,+\infty)$$

are considered in $(L_\infty)_+^n := \left(L_\infty\left(R_+, \sum, \mu\right)\right)_+^n$ and $(L_\infty)_+^m := (L_\infty(R_+, \sigma, \nu))_+^m$, leading to vectors in R_+^n and R_+^m of real n-tuples and m-tuples

respectively. Under certain circumstances (to be explained later) one may so define the neoclassical scalar valued production function $\phi(\bar{x})$, $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$, but the existence of a steady state production correspondence is not obvious. The result of applying $\bar{x}$ need not be confined to the subset of constant rate output vectors. The validity of steady state models of production needs to be examined in the dynamic framework.

CHAPTER 2

DYNAMIC PRODUCTION CORRESPONDENCES

2.1 Dynamic Output Correspondence

The dynamic output correspondence is a mapping

$$x \in (L_\infty)^n_+ \to \mathbb{P}(x) \in 2^{(L_\infty)^m_+} \tag{2.1-1}$$

in which $\mathbb{P}(x)$ is the subset of vectors of output time histories obtainable from the vector x of input time histories.

2.1.1 Axioms on the Output Sets $\mathbb{P}(x)$

(a) No Output from Null Input

$$\mathbb{P}(0) = \{0\} . \tag{$\mathbb{P}$.1}$$

Recall that $0 \iff (||x_i|| = 0 \ \forall i \in \{1,2, \ldots, n\})$.

(b) Boundedness of Output

$$\mathbb{P}(x) \text{ is bounded for } ||x|| < +\infty . \tag{$\mathbb{P}$.2}$$

$$\mathbb{P}(x) \text{ is totally bounded for } ||x|| < +\infty . \tag{$\mathbb{P}$.2S}$$

The first of these two axioms on boundedness merely requires that there exists in the Banach space $(L_\infty)^m$ a ball, centered at 0, of finite radius containing $\mathbb{P}(x)$. In the stronger axiom, there exists for any $\epsilon \in (0,+\infty)$ a finite collection of neighborhoods $N_\epsilon\left(u_o^{\alpha_i}\right)$ centered on the vectors $u_o^{\alpha_i}$, i.e., an ϵ-net, such that

$$\mathbb{P}(x) \subset \bigcup_{i=1}^{K(\epsilon)} N_\epsilon\left(u_o^{\alpha_i}\right)$$

In the metrized Banach space $(L_\infty)^m$, $\rho(u,v) = ||u - v||$, the properties totally bounded and relatively compact are equivalent, which in turn are equivalent to: every infinite sequence $\{u^\alpha\} \subset \mathbb{P}(x)$ contains a Cauchy subsequence. Further, $\mathbb{P}$.2S implies $\mathbb{P}$.2, i.e., a totally bounded output set $\mathbb{P}(x)$ is likewise bounded, but the converse is not true.

The production theoretic significance of $\mathbb{P}(x)$ being totally bounded is that it prohibits infinite time substitution for output histories. Consider the simple case of a single commodity output. Let $\{u^i\}$, $i \varepsilon \{1,2, \ldots\}$, be a sequence of output rate functions defined by

$$u^i(t) = \begin{cases} 1 & \text{if} \quad t \;\varepsilon \left[i, i + \frac{1}{2}\right] \\ 0 & \text{otherwise.} \end{cases} \tag{2.1.1-1}$$

Under the essential sup norm

$$\{u^i\} \subset \left\{f \;\varepsilon\; (L_\infty)^1 : ||f|| \leqq 1\right\} ,$$

and $\mathbb{P}(x) = \{u : u = \theta u^i , i \;\varepsilon\; \{1,2, \ldots\} , \theta \;\varepsilon\; [0,1]\}$ is bounded. But $\mathbb{P}(x)$ is not totally bounded, i.e., it cannot be covered by a finite net of neighborhoods for any $\epsilon \;\varepsilon\; (0,+\infty)$. The sequence $\{u^i\}$ exhibits a substitution of output history which is infinite in time

shift. Positive output can be obtained over any one of the intervals $\left[i, i+\frac{1}{2}\right]$, $(i = 1,2, \ldots)$ as alternatives, i.e., the distributions over time of output are of unbounded extension. In this sense total boundedness of $\mathbb{P}(x)$ prohibits infinite time substitution.

Closure of the subset $\mathbb{P}(x)$ under merely boundedness of $\mathbb{P}(x)$ can lead to compactness provided a weaker topology for $(L_\infty)^m$ is used than the norm topology, for example the weak* topology. Discussion of this alternative is contained in the following section.

(c) Disposability of Inputs

(IP.3) $$\mathbb{P}(\lambda x) \supset \mathbb{P}(x) \quad \text{for} \quad \lambda \in [1,+\infty) .$$

(IP.3S) $$\mathbb{P}(\lambda_1 x_1, \ldots, \lambda_n x_n) \supset \mathbb{P}(x) \ , \ \lambda_i \in [1,+\infty) \ , \ i \in \{1, \ldots, n\} .$$

(IP.3SS) $$\mathbb{P}(x') \supset P(x) \ , \ x' \geqq x .$$

The first of these three properties is a weak assumption for disposal of inputs. It assures that if all input time histories are scaled upward by the same factor, the set of vectors of output histories will not be diminished. The stronger form IP.3S does not require scaling by the same factor, while the superstrong form only requires that each time history of x' be at least as large as the corresponding one for x. Clearly IP.3SS $\Rightarrow$ IP.3S $\Rightarrow$ IP.3.

(d) Existence of Positive Production for all Net Outputs

(IP.4.1) For each output history component i, there exist a vector of input histories $x^{(i)}$ such that $u \in \mathbb{P}(x^{(i)})$ with $u_i \neq 0$.

(e) Attainability of Scaled Outputs

(IP.4.2) If $\bar{u} \in \mathbb{P}(x)$ $||x|| > 0$, $||\bar{u}|| > 0$, there exists for each scalar $\theta \in (0,+\infty)$ a scalar $\lambda_\theta \in (0,+\infty)$ such that $(\theta\bar{u}) \in \mathbb{P}(\lambda_\theta x)$.

In this property nothing is implied about efficiency of attaining $(\theta\bar{u})$ by use of $(\lambda_\theta x)$. It asserts only that replication is possible. Resource limitations are not at issue here. The property is stated for an unconstrained technology. Nothing here implies that inputs are unconstrained, but one cannot develop a general theory in terms of the unlimited details of the latter.

(f) Closure of the Dynamic Correspondence $x \to \mathbb{P}(x)$

(IP.5) $\{x^\alpha\} \to x^o$, $\{u^\alpha\} \to u^o$, $u^\alpha \in \mathbb{P}(x^\alpha)$ for all $\alpha = 1,2, \ldots ,$ imply $u^o \in \mathbb{P}(x^o)$.

In particular the map sets $\mathbb{P}(x)$ are closed for $x \in (L_\infty)^n_+$. Together, IP.2S and IP.5 imply $\mathbb{P}(x)$ is compact.

(f') Upper Semi-Continuity of the Dynamic Correspondence

(IP.5 Bis) For each $x \in (L_\infty)^n_+$ and neighborhood $N(\mathbb{P}(x))$ of $\mathbb{P}(x)$ there exists a neighborhood $N(x)$ such that $\mathbb{P}(z) \subset N(\mathbb{P}(x))$ for $z \in N(x)$.

Here neighborhoods $N(\mathbb{P}(x))$ and $N(x)$ are any subsets of $(L_\infty)^m_+$ and $(L_\infty)^n_+$ respectively containing open sets containing $\mathbb{P}(x)$ and x respectively.

The property $\mathbb{P}.5$ Bis is an additional assumption when $\mathbb{P}.2S$ and $\mathbb{P}.5$ are used under the strong topology for the output space, to guarantee that

$$\mathbb{P}(K) = \{u \mid x \varepsilon K , u \varepsilon \mathbb{P}(x)\}$$

is compact for K compact. See (Berge, 1963), Chapter VI, §1 for proof that the map set of a compact subset K is compact, if the mapping is upper semi-continuous and compact valued. More particularly, in the case of the production correspondence $x \rightarrow \mathbb{P}(x)$ the following theorem holds.

Proposition 2.1.1-1:

$x \rightarrow \mathbb{P}(x)$ is upper semi-continuous under $\mathbb{P}.2S$ and $\mathbb{P}.5$ if and only if $\mathbb{P}(X) = \left\{u \varepsilon (L_\infty)^m_+ : u \varepsilon \mathbb{P}(x) , x \varepsilon X\right\}$ is totally bounded for X totally bounded.

Proof:

Let $x \rightarrow \mathbb{P}(x)$ be upper semi-continuous under $\mathbb{P}.2S$ and $\mathbb{P}.5$, the latter implying that $x \rightarrow \mathbb{P}(x)$ is compact valued. Take X totally bounded, and let $\{u^j\} \subset \mathbb{P}(X)$ be chosen arbitrarily. We need to show that $\{u^j\}$ contains a Cauchy subsequence. Now $\{u^j\} \subset \mathbb{P}(X)$ implies $u^j \varepsilon \mathbb{P}(x^j)$ for $\{x^j\} \subset X$. Since X is totally bounded (by supposition), $\{x^j\}$ contains a Cauchy subsequence $\left\{x^{j_k}\right\}$. Let $\left\{u^{j_k}\right\}$ be the subsequence of $\{u^j\}$ corresponding to $\left\{x^{j_k}\right\}$. Moreover, since $x \rightarrow \mathbb{P}(x)$ is upper semi-continuous, $\left\{u^{j_k}\right\}$ contains a Cauchy subsequence, because $\left\{x^{j_k}\right\}$ is compact, being a subset of X, and $\mathbb{P}\left(\left\{x^{j_k}\right\}\right)$ is compact since $x \rightarrow \mathbb{P}(x)$ is compact

valued. Thus $\mathbb{P}(X)$ is totally bounded. Conversely assume $\mathbb{P}(X)$ is totally bounded for X totally bounded, and consider $u^j \in \mathbb{P}(x^j)$ for $\{x^j\} \subset X$. For X totally bounded $\{x^j\}$ is a totally bounded subset of X and $\{u^j\} \subset \mathbb{P}(\{x^j\})$ is totally bounded. Then $\{u^j\}$ contains a Cauchy subsequence converging to $u^o \in \mathbb{P}(x^o)$ for $\{x^j\} \to x^o$, since $x \to \mathbb{P}(x)$ satisfies $\mathbb{P}.5$. Consequently $x \to \mathbb{P}(x)$ is upper semi-continuous. [See (Hildenbrand, 1974), Theorem 1, §BIII.]

(g) Disposability of Outputs

($\mathbb{P}$.6) If $u \in \mathbb{P}(x)$, $(\theta u) \in \mathbb{P}(x)$, $\theta \in [0,1]$.

($\mathbb{P}$.6S) If $u \in \mathbb{P}(x)$, $(\theta_1 u_1, \theta_2 u_2, \ldots, \theta_m u_m) \in \mathbb{P}(x)$, $\theta_i \in [0,1]$, $i \in \{1,2,\ldots,m\}$.

($\mathbb{P}$.6SS) If $u \in \mathbb{P}(x)$, $\{u : 0 \leqq v \leqq u\} \subset \mathbb{P}(x)$.

The first of these three properties is a weak assumption for disposal of outputs, asserting that, if all time histories of an attainable vector of the same are scaled downward by the same factor, the result will also be obtainable. The stronger form $\mathbb{P}$.6S does not limit scaling only to the same factor and the last form $\mathbb{P}$.6SS does not even require congruence of the time distributions of the output time histories. Clearly $\mathbb{P}.6SS \Rightarrow \mathbb{P}.6S \Rightarrow \mathbb{P}.6$.

As additional properties not generally used one may include:

(h) Convexity of Output Sets

($\mathbb{P}$.7) The correspondence $x \to \mathbb{P}(x)$ is quasi-concave, i.e., $\mathbb{P}((1-\lambda)x + \lambda y) \supset \mathbb{P}(x) \cap \mathbb{P}(y)$ for $\lambda \in [0,1]$.

$$(\mathbb{P}.8) \qquad \begin{array}{l}\text{The set } \mathbb{P}(x) \text{ of output histories} \\ \text{are convex, for } x \in (L_\infty)^n_+ \ .\end{array}$$

It is clear that $\mathbb{P}.1$ and any of the three forms of $\mathbb{P}.6$ imply that the vector of null output time histories, i.e., 0, belongs to all output sets $\mathbb{P}(x)$, $x \in (L_\infty)^n_+$ i.e., it is attainable under all circumstances.

Any disposal of outputs, except scaling downward by a common factor, implies absence of restriction to decrease and/or discard unwanted outputs. Since the model of production considered is a technical statement, including all components of the vector u whether wanted or not, whether harmful or not, an assumption of disposability is a socio-economic premise and not strictly a matter of the technology of production. Scaling downward by a common factor is a reduction of scale of operation, which is possible on purely technical grounds.

The disposal allowed by $\mathbb{P}.6S$, like $\mathbb{P}.6$, does not change the time support of an output history to transfer the impact of unwanted outputs to future generations by free disposal in time, like $\mathbb{P}.6SS$. However, the mix of output histories can be altered.

The foregoing list of properties for the output correspondence $x \to \mathbb{P}(x)$ is not intended as a complete, independent set of axioms. Alternatives are included for subsequent use and discussion. Also further axioms will be made subsequently on the time distribution of input and output histories.

2.1.2 Weak Topology for the Output Space

Recall that the sets $\mathbb{P}(x)$ of vectors of output histories are elements of the power set of the Banach space $(L_\infty)^m$ with norm

$$||f|| = \underset{i}{\text{Max}}\, ||f_i|| \ , \ f = (f_1, f_2, \ldots, f_m) \ \varepsilon \ (L_\infty)^m \ ,$$

$$||f_i|| = \nu_i\text{-Essential} \ \underset{t}{\text{Sup}} \ |f_i(t)| \ , \ t \ \varepsilon \ [0,+\infty) \ ,$$

and metric $\rho(f,g) = ||f - g||$. The open sets of the topology of $(L_\infty)^m$ were taken as the open spheres defined by this metric, and in this topology bounded and closed output history sets $\mathbb{P}(x)$ need not be compact. For this reason, property $\mathbb{P}$.2S (total boundedness) was included in the list as an alternative with $\mathbb{P}$.5 to obtain a compact valued dynamic correspondence.

The weaker assumption $\mathbb{P}$.2, requiring only that $\mathbb{P}(x)$ be bounded yields a compact valued correspondence when taken with property $\mathbb{P}$.5 (closure), provided a weaker topology is used for $(L_\infty)^m$. This weaker topology (weak star) is constructed as follows: Let r_i $(i = 1,2, \ldots, m)$ be a real valued function defined on R_+ corresponding to the output history (function) u_i . Symbolically:

$$r_i : r_i(t) \ , \ t \ \varepsilon \ [0,+\infty) \ , \ i \ \varepsilon \ \{1,2, \ldots, m\} \ . \tag{2.1.2-1}$$

The function r_i are restricted to those which are summable in absolute value on R_+ , i.e.

$$\int_0^\infty |r_i(t)|d\nu_i(t) < +\infty \ , \ i \ \varepsilon \ \{1,2, \ldots, m\}$$

with respect to a sigma-finite measure space (R_+, σ_i, ν_i). Thus for the i^{th} net output we consider a function element r_i of a space $L_1(R_+, \sigma_i, \nu_i)$ of ν_i-Equivalence Classes, normed by

$$(2.1.2\text{-}2) \qquad ||r_i|| = \int_0^\infty |r_i(t)| d\nu_i(t) .$$

Together the functions r_i, define a vector $r = (r_1, r_2, \ldots, r_m)$ which is an element of the product space $(L_1)^m := \mathop{X}\limits_{i=1}^{m} L_1(R_+, \sigma_i, \nu_i)$ of absolute value summable equivalence classes. The norm of r may be defined by

$$(2.1.2\text{-}3) \qquad ||r|| = \mathop{\mathrm{Max}}_i ||r_i|| , \; i \in \{1, 2, \ldots, m\} .$$

The dual or conjugate space $L_1^*(R_+, \sigma_i, \nu_i)$ is isometrically isomorphic to the space $L_\infty(R_+, \sigma_i, \nu_i)$ of output functions u_i for the i^{th} exogenous output. A weak* topology is defined for $L_\infty(R_+, \sigma_i, \nu_i)$ by a base of neighborhoods U_i defined at 0 and the null output history of equivalence class, consisting of the open sets

$$(2.1.2\text{-}4) \qquad \left\{ u_i \in (L_\infty)_i : \sup_{\alpha=1,2,\ldots,k} \int_0^\infty |r_i^\alpha(t) u_i(t) d\nu_i(t)| < \epsilon \right\}$$

for $k \in \{1, 2, \ldots\}$, where $\left\{ r_i^1, r_i^2, \ldots, r_i^k \right\}$ is a finite collection of elements from the space $L_1(R_+, \sigma_i, \nu_i)$. The weak* topology for

$$(L_\infty)^m := \mathop{X}\limits_{i=1}^{m} L_\infty(R_+, \sigma_i, \nu_i)$$

has a neighborhood base at 0 given by

$$U = \overset{m}{\underset{1}{X}} U_i \,, \tag{2.1.2-5}$$

where $\overset{m}{\underset{1}{X}}$ denotes the set of all vectors $(u_1, u_2, \ldots, u_m)$ such that $u_i \in U_i$.

In this construction the norms $||u_i||$ may be taken as

$$||u_i|| = \underset{||r_i|| \leqq 1}{\text{Sup}} \left(\int_0^\infty |r_i(t) u_i(t) d\nu_i(t)| \right) \tag{2.1.2-6}$$

in place of 1-15 with $||u||$ defined by 1-16.

As a consequence of the Alaoglu-Bourbaki theorem on neighborhoods of 0 in a locally convex space [see (Köthe, 1969), §20, 10], the closed unit ball of the dual of a normed space is weak* compact. Accordingly, since $\mathbb{P}(x)$ is bounded it is weak* relatively compact. Then property $\mathbb{P}.5$ taken in the weak* topology implies that $\mathbb{P}(x)$ is weak* compact.

The weak* topology for $L_\infty(R_+, \sigma_i, \nu_i)$ is the coursest topology for the linear functional

$$\langle u_i, r_i \rangle = \int_0^\infty r_i(t) u_i(t) d\nu_i(t) \tag{2.1.2-7}$$

to be continuous in u_i for each element r_i of $L_1(R_+, \sigma_i, \nu_i)$ in the real field.

The weak* topology considered here arises for all dual pairs of topological vector spaces (E_2, E_1) when an element of the real field is associated with every pair $(f,g) \in E_2 \times E_1$ through a bilinear form $\langle f,g \rangle$ where: $g \in E_1$ and $\langle f,g \rangle = 0 \;\forall f \in E_2 \Rightarrow g = 0$, and $f \in E_2$

and $<f,g> = 0 \ \forall g \in E_1 \Rightarrow f = 0$. In case the dual pair is of the form (E^*,E) , where E^* is the dual or conjugate space of E , as used here, the topology is referred to as the weak* (weak star) topology.

In economic theory the spaces $L_1(R_+,\sigma_i,\nu_i)$ used for the construction of the weak* topology have a natural economic interpretation. Each function r_i may be regarded as a price history for the i^{th} exogenous output which is summable in absolute value. Negative values $r_i(t)$ in this history can be meaningful, when the i^{th} exogenous output is unwanted. In order to have the production correspondence as a complete technical statement and for obvious social purposes in production planning, unwanted outputs are not deleted from the list of outputs of a production technology. Thus the functions r_i are not restricted to the nonnegative domains $(L_1(R_+,\sigma_i,\nu_i))_+$.

The norms (2.1.2-5), (1-16) suggested for vectors of output functions u provide a comparison in terms of the entire output histories u_i as opposed to (1-15), (1-16) which involve only the supremal values of these histories. If two vectors u and v differ only by one having zero output history for certain unwanted outputs, the vector with positive unwanted output rates will have larger, norm, but would not be preferred. The norms for these abstract spaces cannot make such distinctions, which are best done in the context of an economic process of evaluation.

It should be noted that in the weak* topology $\{x^j\} \to \{x^o\}$, $u^j \in \mathbb{P}(x^j)$ for $j = 1,2, \ldots$, and $\{u^j\} \to u^o$ does not imply $u^o \in \mathbb{P}(x^o)$ if property $\mathbb{P}$.5 is stated in the strong topology.

Accordingly, when $\mathbb{P}.2$ is used with the weak* topology, property $\mathbb{P}.5$ will be taken in the weak* topology and designated $\mathbb{P}.5.^*$

2.1.3 Axioms on Time Spans of Outputs

In Section 2.1.1 certain axioms were stated for the dynamic correspondence $x \to \mathbb{P}(x)$ which related to time histories of outputs as primitive elements, with no characterizations for time distribution of these histories. In production, inputs do not instantaneously yield outputs, i.e., it takes time for output to be realized, and some additional axiom is needed for the dynamic correspondence to reflect this fact. Also, if inputs are altogether stopped in application, outputs cannot continue and an axiom is needed to reflect this fact.

Let $f = (f_1, f_2, \ldots, f_k) \in (L_\infty)^k_+$ and define:

$$S_i := \{t \in R_+ : f_i(t) > 0\}, \; i \in \{1,2, \ldots, k\} \tag{2.1.3-1}$$

$$\bar{t}_f := \underset{i}{\text{Max}} \text{ Ess Sup } \{t : t \in S_i, \; i \in \{1,2, \ldots, k\}\} \tag{2.1.3-2}$$

$$\underline{t}_f := \underset{i}{\text{Min}} \text{ Ess Inf } \{t : t \in S_i, \; i \in \{1,2, \ldots, k\}\} \tag{2.1.3-3}$$

where

$$\begin{aligned} &\text{Ess Sup } \{t : t \in S_i\} := \text{Inf } \{\alpha \in R_+ : \text{Meas } (\{t \in S_i : t > \alpha\}) = 0\} \\ &\text{Ess Inf } \{t : t \in S_i\} := \text{Sup } \{\alpha \in R_+ : \text{Meas } (\{t \in S_i : t < \alpha\}) = 0\} . \end{aligned} \tag{2.1.3-4}$$

(a) Axiom on Initiation of Output

$$(\mathbb{P}.T.1) \qquad \underline{t}_u > \underline{t}_x \; , \; x \in (L_\infty)^n_+ \; , \; u \in \mathbb{P}(x) \subset (L_\infty)^m_+ \; .$$

(b) Axiom on Time Extension of Output

$$(\mathbb{P}.T.2) \qquad \bar{t}_u \leqq \bar{t}_x \; , \; x \in (L_\infty)^n_+ \; , \; u \in \mathbb{P}(x) \subset (L_\infty)^m_+ \; .$$

Axiom ($\mathbb{P}$.T.1) states that a positive interval of time must elapse after initiation of inputs, before outputs can begin to issue. Here "initiation" and "issue" refer to positive time rates of some input and some output, respectively, over time intervals of positive measure.

When all inputs are no longer applied at positive rate over time intervals of positive measure, outputs must cease. Work in progress cannot be completed without some exogenous input rates applied over intervals of positive measure. The inequality sign is used in $\mathbb{P}$.T.2, because of the possible absence of some inputs necessary for positive outputs, in the vector x of time histories of inputs.

2.2 Dynamic Input Correspondence

The dynamic input correspondence is a mapping

$$(2.2\text{-}1) \qquad u \in (L_\infty)^m_+ \rightarrow \mathbb{L}(u) \in 2^{(L_\infty)^n_+}$$

in which $\mathbb{L}(u)$ is the subset of input histories defined by

$$(2.2\text{-}2) \qquad \mathbb{L}(u) := \left\{ x \in (L_\infty)^n_+ : u \in \mathbb{P}(x) \right\} \; .$$

The input and output dynamic correspondences are inversely related by:

(2.2-3) $$x \in \mathbb{L}(u) \Longleftrightarrow u \in \mathbb{P}(x) .$$

Consequently the axioms for $x \to \mathbb{P}(x)$ imply certain properties for $u \to \mathbb{L}(u)$, and conversely.

2.2.1 Properties Induced for $u \to \mathbb{L}(u)$

(a) All Inputs Yield at Least No Ouput and No Input Yields No Output

($\mathbb{L}$.1) $$\mathbb{L}(0) = (L_\infty)^n_+ , \; 0 \notin \mathbb{L}(u) \quad \text{for} \quad ||u|| > 0 .$$

The axioms $\mathbb{P}$.1 and $\mathbb{P}$.6 imply $0 \in \mathbb{P}(x)$ for all $x \in (L_\infty)^n_+$, and by (2.2-2) $\mathbb{L}$.1 follows. When $||u|| > 0$, either $\mathbb{L}(u)$ is empty with $0 \notin \mathbb{L}(u)$, or not empty with $0 \in \mathbb{L}(u)$ not possible, because then $u \in \mathbb{P}(0)$ contradicting $\mathbb{P}$.1.

(b) Intersection of Output Sequences

($\mathbb{L}$.2) If $||u^\alpha|| \to +\infty$, $\alpha = 1,2, \ldots,$ $\bigcap_{\alpha=1}^{\infty} \mathbb{L}(u^\alpha)$ is empty.

($\mathbb{L}$.2S) If $||x|| < +\infty$ and $x \in \mathbb{L}(u^\alpha)$, $\alpha = 1,2, \ldots,$ $\{u^\alpha\}$ contains a Cauchy subsequence.

Property $\mathbb{L}$.2 is a direct consequence of $\mathbb{P}$.2, because in the contrary case $x \in \bigcap_{\alpha=1}^{\infty} \mathbb{L}(u^\alpha)$ and consequently $x \in \mathbb{L}(u^\alpha)$ for all $\alpha = 1,2, \ldots$ with $||u^\alpha|| \to +\infty$, contradicting $\mathbb{P}$.2.

Property $\mathbb{L}.2S$ is a direct consequence of $\mathbb{P}.2S$, because $\mathbb{P}(x)$ is totally bounded if and only if each infinite sequence contained in $\mathbb{P}(x)$, i.e., $\{u^\alpha\} \subset \mathbb{P}(x)$, has a Cauchy subsequence, and inversely $||x|| < +\infty$, $x \in \mathbb{L}(u^\alpha)$, with $\{u^\alpha\}$ having a Cauchy subsequence.

Moreover, the stronger property $\mathbb{L}.2S$ implies $\mathbb{L}.2$, but the converse does not hold. The example of §2.1.1 shows that $\mathbb{P}.2$ does not imply $\mathbb{P}.2S$, and hence $\mathbb{L}.2$ does not imply $\mathbb{L}.2S$.

If the weak* topology is used for the output space $(L_\infty)_+^m$, the property $\mathbb{P}.2$ (boundedness) implies that $\mathbb{P}(x)$ is relatively compact, or equivalently totally bounded in this Banach space of vectors of output functions. Then property $\mathbb{L}.2$, which is equivalent to $\mathbb{P}.2$, implies $\mathbb{L}.2S$ and properties $\mathbb{L}.2$ and $\mathbb{L}.2S$ are equivalent. Properties $\mathbb{P}.2$ and $\mathbb{P}.2S$ are equivalent since a totally bounded subset of a metric space is bounded.

(c) Disposability of Inputs

($\mathbb{L}.3$) If $x \in \mathbb{L}(u)$, $(\lambda x) \in \mathbb{L}(u)$ for $\lambda \in [0,+\infty)$.

($\mathbb{L}.3S$) If $x \in \mathbb{L}(u)$, $(\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_n x_n) \in \mathbb{L}(u)$ for $\lambda_i \in [1,+\infty)$, $i \in \{1,2, \ldots, n\}$.

($\mathbb{L}.3SS$) If $x \in \mathbb{L}(u)$ and $x' \geqq x$, $x' \in \mathbb{L}(u)$.

These three properties are a direct consequence of $\mathbb{P}.3$, $\mathbb{P}.3S$, and $\mathbb{P}.3SS$ respectively for the output correspondence, obtained merely by inversing these statements. For example: $x \in \mathbb{L}(u) \Longleftrightarrow u \in \mathbb{P}(x) \subset \mathbb{P}(\lambda x)$, $\lambda \in [1,+\infty)$ and $u \in \mathbb{P}(\lambda x) \Longleftrightarrow (\lambda x) \in \mathbb{L}(u)$ for $\lambda \in [1,+\infty)$.

Property $\mathbb{L}.3$ states merely that, if a vector x of input histories can obtain a given vector u of output histories, any common scaling upward of the individual input histories of x can still obtain u .

Property $\mathbb{L}.3S$ asserts that in the scaling upward of the input histories x_i a common factor need not be applied, but as a special case they can be the same, i.e., $\mathbb{L}.3S \Rightarrow \mathbb{L}.3$.

Property $\mathbb{L}.3SS$ does not even require the individual input histories to be increased over time by a common factor, i.e., a history x_i may be increased in any fashion over a subset of R_+ of positive measure.

For future discussion $\mathbb{L}.3 \Longleftrightarrow \mathbb{P}.3$ will be referred to as Weak Disposal of inputs, $\mathbb{L}.3S \Longleftrightarrow \mathbb{P}.3S$ will be referred to as Strong Disposal of inputs and $\mathbb{L}.3SS \Longleftrightarrow \mathbb{P}.3SS$ will be called Free Disposal of inputs.

(d) Possibility of $||u_i|| > 0$, $i \varepsilon \{1,2, \ldots, m\}$

$$\left\{x \ \varepsilon \ (L_\infty)^n_+ : ||u_i|| > 0 \ \text{ for some } \ u \ \varepsilon \ \mathbb{P}(x)\right\} \neq \emptyset \quad i \ \varepsilon \ \{1,2, \ldots, m\} \ . \tag{$\mathbb{L}.4.1$}$$

(e) Scaling of Inputs

$$\begin{aligned} &\text{If} \quad (\bar{\lambda}x) \ \varepsilon \ \mathbb{L}(\bar{u}) \ , \ ||x|| > 0 \ , \ ||\bar{u}|| > 0 \ , \ \bar{\lambda} \, \varepsilon \, (0,+\infty) \\ &\{\lambda x : \lambda \, \varepsilon \, [0,+\infty)\} \cap \mathbb{L}(\theta \bar{u}) \neq \emptyset \quad \text{for} \quad \theta \ \varepsilon \ [0,+\infty) \ . \end{aligned} \tag{$\mathbb{L}.4.2$}$$

(f) Closure of the Dynamic Correspondence $u \to \mathbb{L}(u)$

$$\begin{aligned} &\{x^\alpha\} \to x^o \ , \ \{u^\alpha\} \to u^o \ , \ x^\alpha \ \varepsilon \ \mathbb{L}(u^\alpha) \quad \text{for} \\ &\text{all} \quad \alpha = 1,2, \ \ldots \ , \ \text{imply} \quad x^o \ \varepsilon \ \mathbb{L}(u^o) \ . \end{aligned} \tag{$\mathbb{L}.5$}$$

The properties $\mathbb{L}.4.1$, $\mathbb{L}.4.2$ and $\mathbb{L}.5$ are merely restatements of $\mathbb{P}.4.1$, $\mathbb{P}.4.2$ and $\mathbb{P}.5$ respectively. Property $\mathbb{L}.5$ implies that the sets $\mathbb{L}(u)$ of vectors of input functions are closed.

(g) Nondecreasing Inputs for Scaling Outputs

$$(\mathbb{L}.6) \qquad \mathbb{L}(\theta u) \subset \mathbb{L}(u) \ , \ \theta \in [1,+\infty) \ .$$

$$(\mathbb{L}.6S) \qquad \begin{gathered} \mathbb{L}(\theta_1 u_1, \theta_2 u_2, \ldots, \theta_m u_m) \subset \mathbb{L}(u) \ , \\ \theta_i \in [1,+\infty) \ , \ i \in \{1,2, \ldots, m\} \ . \end{gathered}$$

$$(\mathbb{L}.6SS) \qquad \mathbb{L}(u') \subset \mathbb{L}(u) \ , \ u' \geqq u \ .$$

These three properties follow directly from properties $\mathbb{P}.6$, *$\mathbb{P}.6S$, $\mathbb{P}.6SS$ respectively.* Consider $\theta_i \geqq 1$, $i \in \{1,2, \ldots, m\}$ and let $x \in \mathbb{L}(\theta_1 u_1, \theta_2 u_2, \ldots, \theta_m u_m)$ when this input set is non-empty. Then inversely $(\theta_1 u_1, \theta_2 u_2, \ldots, \theta_m u_m) \in \mathbb{P}(x)$ and for $\theta_i = 1$, $i \in \{1,2, \ldots, m\}$, $u \in \mathbb{P}(x)$, implying inversely that $x \in \mathbb{L}(u)$. If the set $\mathbb{L}(\theta_1 u_1, \ldots, \theta_m u_m)$ is empty, $\mathbb{L}.6S$ is trivially satisfied. Property $\mathbb{L}.6$ follows in case $\theta_i = \theta$, $i \in \{1,2, \ldots, m\}$. When $\mathbb{P}.6SS$ holds, $x \in \mathbb{L}(u')$, $u' \geqq u$, implies $u \leqq u' \in \mathbb{P}(x)$ and $x \in \mathbb{L}(u)$. Thus $\mathbb{L}.6SS$ is implied.

By the arguments given above, it is clear that ($\mathbb{L}.1$, $\mathbb{L}.2$, $\mathbb{L}.3$, $\mathbb{L}.4.1$, $\mathbb{L}.4.2$, $\mathbb{L}.5$ and $\mathbb{L}.6$) imply ($\mathbb{P}.1$, $\mathbb{P}.2$, $\mathbb{P}.3$, $\mathbb{P}.4.1$, $\mathbb{P}.4.2$, $\mathbb{P}.5$ and $\mathbb{P}.6$), and the same holds for any substitution of ($\mathbb{L}.3$, $\mathbb{L}.6$) by stronger forms with the corresponding strong forms of ($\mathbb{P}.3$, $\mathbb{P}.6$) used.

Accordingly one may start either with the correspondence (2.2-1) or (2.1-1) and related properties as axioms, and deduce the corresponding properties for the other.

The additional properties $\mathbb{P}.7$, $\mathbb{P}.8$ for the output correspondence $x \to \mathbb{P}(x)$ imply the following for the input correspondence $u \to \mathbb{L}(u)$ and conversely:

($\mathbb{L}.7$) The sets $\mathbb{L}(u)$ of vectors of input functions are convex.

($\mathbb{L}.8$) The input correspondence $u \to \mathbb{L}(u)$ is Quasi-Concave, i.e., $\mathbb{L}((1-\theta)u + \theta v) \supset \mathbb{L}(u) \cap \mathbb{L}(v)$ for $\theta \in [0,1]$.

This discussion may be summarized by the following proposition.

Proposition 2.2.1-1:

If $x \to \mathbb{P}(x)$ and $u \to \mathbb{L}(u)$ are inversely related dynamic production correspondences, $\{\mathbb{P}.1, \mathbb{P}.2, \mathbb{P}.3, \mathbb{P}.4.1, \mathbb{P}.4.2, \mathbb{P}.5, \mathbb{P}.6\} \Leftrightarrow \{\mathbb{L}.1, \mathbb{L}.2, \mathbb{L}.3, \mathbb{L}.4.1, \mathbb{L}.4.2, \mathbb{L}.5, \mathbb{L}.6\}$, with corresponding stronger forms substituted for $\mathbb{P}.3$, $\mathbb{P}.6$ and $\mathbb{L}.3$, $\mathbb{L}.6$. The properties $\mathbb{P}.2S$ and $\mathbb{L}.2S$ may be substituted for $\mathbb{P}.2$ and $\mathbb{L}.2$ under the weak* topologies for $(L_\infty)^m_+$ and $(L_\infty)^n_+$.

2.2.2 Axioms on Time Extension of Inputs

($\mathbb{L}.T.1$) (a) If $u \in (L_\infty)^m_+$, u is summable on $[0,+\infty)$ in each component and $\mathbb{L}(u)$ is not empty, there exists a vector $x \in (L_\infty)^n_+$ of summable input histories x_i on $[0,+\infty)$ such that $x \in \mathbb{L}(u)$.

(b) If $\bar{t}_u < +\infty$, $\mathbb{L}(u)$ is not empty and $x \in \mathbb{L}(u)$, then the vector

($\mathbb{L}.T.2$)
$$y := \begin{Bmatrix} y_i(t) = x_i(t) \ , \ t \in [0,\bar{t}_u] \\ y_i(t) = 0 \ , \ t \in (\bar{t}_u,+\infty) \end{Bmatrix} , \ i \in \{1,2, \ldots, n\} ,$$

of input histories also belongs to $\mathbb{L}(u)$.

The first axiom guarantees that if outputs are summable on $[0,+\infty)$, they may be obtained by a vector of input histories whose components are summable on $[0,+\infty)$. The second axiom is stronger, stating that if the support of the vector u of output histories is a bounded subinterval of R_+, inputs beyond the upper limit of this interval are not required for the given output histories. With this axiom, any restoring of facilities or environment required with a given production are considered as part of the output.

2.2.3 Technically Efficient Input Histories

Efficient input histories for a vector u of output histories are denoted by a subset $\mathbb{E}(u)$ of $\mathbb{L}(u)$ and defined by

Definition (2.2.3-1):

$x \in \mathbb{E}(u)$ iff $x \in \mathbb{L}(u)$ and $y \notin \mathbb{L}(u)$ for $y \leq x$, $u \geq 0$ and $\mathbb{L}(u)$ not empty. $\mathbb{E}(0) := \{0\}$.

The efficiency so defined is a technical property of being a vector x of input histories such that no decrease is possible for any of the histories x_i on a subset of $[0,+\infty)$ of positive measure, without increasing that of some other input history x_j on a subset of $[0,+\infty)$ of positive measure, in order to still attain the vector u of output histories.

It will be useful for subsequent considerations to use the notation

$$(2.2.3\text{-}1)\qquad (\tilde{L}_\infty)_+^\alpha := \left\{z \in (L_\infty)_+^\alpha : z_i \text{ is summable on } [0,+\infty),\ i \in \{1, \dots, \alpha\}\right\}.$$

$$(2.2.3\text{-}2)\qquad \tilde{\mathbb{L}}(u) := \left\{x \in (L_\infty)_+^n : x \in \mathbb{L}(u) \cap (\tilde{L}_\infty)_+^n\right\}.$$

(2.2.3-3) $\tilde{\mathbb{E}}(u) := \left\{ x \in (L_\infty)^n_+ : x \in \tilde{\mathbb{L}}(u) \ , \ y \leq x \Rightarrow y \notin \tilde{\mathbb{L}}(u) \right\}$.

The following proposition holds:

Proposition (2.2.3-1):

If $u \in (\tilde{L}_\infty)^m_+$ and $\mathbb{L}(u)$ is not empty, then $\tilde{\mathbb{E}}(u)$ is not empty and $\tilde{\mathbb{E}}(u) \subset \mathbb{E}(u)$.

By the axiom $\mathbb{L}$.T.1 there exists $y \in \mathbb{L}(u)$ such that $y \in (\tilde{L}_\infty)^n_+$. For $x \in A(y) := \mathbb{L}(u) \cap \left\{ x \in (L_\infty)^n_+ : x \leqq y \right\}$ consider the functional

$$F(x) := \int_0^\infty \sum_{i=1}^n x_i(t) d\mu_i(t) \ . \tag{2.2.3-4}$$

Now $\operatorname{Min} \{F(x) : x \in A(y)\} = F(x^*)$ exists for some $x^* \in \tilde{D}(y)$, since $\tilde{D}(y)$ is closed and each of the functions x_i are nonnegative and bounded below by $0 \in (L_\infty)_+$. Further $x^* \in \mathbb{L}(u)$, since $x^* \in \mathbb{L}(u) \cap (\tilde{L}_\infty)^n_+$ and, if $y \in \tilde{\mathbb{L}}(u)$ with $y \leq x^*$, then $F(y) < F(x^*)$ a contradiction. Finally $\tilde{\mathbb{E}}(u) \subset \mathbb{E}(u)$, because if $x \in \tilde{\mathbb{E}}(u)$, then $x \in \mathbb{L}(u)$ and $y \leq x$ with $y \in \mathbb{L}(u)$ is impossible since y has to belong to $(\tilde{L}_\infty)^n_+$.

Proposition (2.2.3-1):

If $\bar{t}_u < +\infty$ and $\mathbb{L}(u)$ is not empty, $\tilde{\mathbb{E}}(u) \subset \left\{ x \in (\tilde{L}_\infty)^n_+ : \bar{t}_x = \bar{t}_u \right\}$.

When $\bar{t}_u < +\infty$, $u \in (\tilde{L}_\infty)^n_+$, and the proposition follows directly from the definition of $\tilde{\mathbb{E}}(u)$ and the axioms $\mathbb{P}$.T.2 and $\mathbb{L}$.T.2.

By a transfinite process the existence of a vector x of input functions belonging to $\mathbb{E}(u)$ for $u \varepsilon (L_\infty)^m_+$, not necessarily with summable components u_i, is shown by the following proposition

Proposition (2.2.3-3):

If $\mathbb{L}(u)$ is not empty, $\mathbb{E}(u)$ is not empty.

Proof:

We need only consider $u \geq 0$, since $\mathbb{E}(0) := \{0\}$. Let $x^o \varepsilon \mathbb{L}(u)$, $u \geq 0$. Define a set F_0 by

$$F_0 := \left(\mathbb{L}(u) \cap \left\{x \varepsilon (L_\infty)^n_+ : x \leqq x^o\right\}\right) \tag{2.2.3-2}$$

and consider

$$f(x^o) := \operatorname*{Sup}_x \left\{\sum_{i=1}^n ||x_i^o - x_i|| : x \varepsilon F_0\right\}. \tag{2.2.3-3}$$

If $f(x^o) = 0$, $x^o \varepsilon \mathbb{E}(u)$. Contrarywise, assume $f(x^o) > 0$ and let x^* be an input history such that $f(x^o) = \sum_1^n ||x_i^o - x_i^*||$. Define $x^1 = x^* + \frac{x^o - x^*}{2} = \frac{x^* + x^o}{2}$ and consider

$$f(x^1) = \operatorname*{Sup}_x \left\{\sum_{i=1}^n ||x_i^1 - x_i|| : x \varepsilon F_1\right\} \tag{2.2.3-4}$$

where F_1 is a closed subset of F_0 defined by

$$F_1 := \left(\mathbb{L}(u) \cap \left\{x \varepsilon (L_\infty)^n_+ : x \leqq x^1\right\}\right). \tag{2.2.3-5}$$

As a first step the following equality is shown to hold.

$$(2.2.3\text{-}6) \quad f(x^1) = \sup_{y,x} \left\{ \sum_{i=1}^{n} ||x_i - y_i|| : x , y \in F_1 \right\} = : \delta(F_1) .$$

Let x^{**} , y^{**} yield $f(x^1)$, i.e., $\delta(F_1) = \sum_1^n ||x_i^{**} - y_i^{**}||$. By the definition of F_1 , $x^1 \geqq x^{**}$; $x^1 \geqq y^{**}$. Consequently

$$\text{Max} \left\{ \sum_1^n ||x_i^1 - x_i^{**}|| , \sum_1^n ||x_i^1 - y_i^{**}|| \right\} \geqq \sum_1^n ||x_i^{**} - y_i^{**}||$$

and $f(x^1) \geqq \delta(F_1)$. But the equality sign holds since $x^1 \in F_1$. From equation (2.2.3-6) it is clear that

$$(2.2.3\text{-}7) \qquad f(x^1) \geqq \sup_{y,x} \{||x - y|| : x , y \in F_1\} ,$$

i.e., $f(x^1)$ is equal to or greater than the maximum diameter of F_1 .

Next it is shown that

$$(2.2.3\text{-}8) \qquad f(x^1) = \sum_{i=1}^{n} ||x_i^1 - x_i^*|| ,$$

where x^* is the vector yielding $f(x^o)$. Observe first (see definition of x_i^1) that

$$\sum_{i=1}^{n} ||x_i^1 - x_i^*|| + \sum_{i=1}^{n} ||x_i^o - x_i^1|| = \sum_{i=1}^{n} ||\frac{x_i^*}{2} + \frac{x_i^o}{2} - x_i^*|| + \sum_{i=1}^{n} ||x_i^o - \frac{x_i^*}{2} - \frac{x_i^o}{2}||$$

$$= \frac{1}{2} \sum_{i=1}^{n} ||x_i^o - x_i^*|| + \frac{1}{2} \sum_{i=1}^{n} ||x_i^o - x_i^*|| = \sum_{i=1}^{n} ||x_i^o - x_i^*|| .$$

Hence

$$(2.2.3\text{-}9) \quad \sum_{i=1}^{n} ||x_i^1 - x_i^*|| + \sum_{i=1}^{n} ||x_i^o - x_i^1|| = \sum_{i=1}^{n} ||x_i^o - x_i^*|| \ .$$

To show (2.2.3-8), let x^{**} yield (2.2.3-4) and assume

$$\sum_{i=1}^{n} ||x_i^1 - x_i^{**}|| > \sum_{i=1}^{n} ||x_i^1 - x_i^*|| \ .$$

Then, using (2.2.3-9),

$$\sum_{i=1}^{n} ||x_i^1 - x_i^{**}|| + \sum_{i=1}^{n} ||x_i^o - x_i^1|| > \sum_{i=1}^{n} ||x_i^o - x_i^*|| + \sum_{i=1}^{n} ||x_i^o - x_i^1|| \ .$$

But, by (2.2.3-3), $\sum_{i=1}^{n} ||x_i^o - x_i^*|| \geqq \sum_{i=1}^{n} ||x_i^o - x_i^{**}||$. Then

$$\sum_{i=1}^{n} ||x_i^1 - x_i^{**}|| + \sum_{i=1}^{n} ||x_i^o - x_i^1|| > \sum_{i=1}^{n} ||x_i^o - x_i^{**}|| + \sum_{i=1}^{n} ||x_i^o - x_i^1|| \ ,$$

and since $\sum_{i=1}^{n} ||x_i^o - x_i^{**}|| = \sum_{i=1}^{n} ||x_i^1 - x_i^{**} + x_i^o - x_i^1|| \geqq \sum_{i=1}^{n} ||x_i^1 - x_i^{**}||$,
a contradiction is obtained. Thus

$$\sum_{i=1}^{n} ||x_i^1 - x_i^{**}|| \leqq \sum_{i=1}^{n} ||x_i^1 - x_i^*|| \ ,$$

and, since $x^* \varepsilon F_1$, (2.2.3-8) is shown to hold.

By proceeding in this manner, letting

$$x^N = x^* + \frac{x^o - x^*}{2^N} \ , \ F_N = \left(\mathbb{L}(u) \cap \left\{x \in (L_\infty)^n_+ : x \leqq x^N\right\}\right)$$

one obtains $F_N \subset F_{N+1}$, $N = 1,2, \ldots$ with $f(x^N) \geqq$ Max. Diam of F_N and $f(x^N) \downarrow 0$ for $N \to \infty$. Then, by the Cantor Intersection Theorem

$$\bigcap_{N=1}^{\infty} F_N = \{x^*\} \ , \ x^* \in \mathbb{L}(u)$$

and $x^* \in \mathbb{E}(u)$.

It should be noted, perhaps, that the axiom $\mathbb{L}$.T.2 does not exclude efficient vectors x for $u \in (\tilde{L}_\infty)^m_+$ which do not have summable component histories x_i . Indeed, if the measures of the supports of the output rate histories u_i are not bounded, one could expect that a nonsummable history x_i can arise, as in the case of certain service inputs. However, even if the supports of the histories u_i had bounded measure, the same situation might arise, because when output rate is zero, some positive input may be required as a standby factor. Withal, almost all cases of interest are those in which the supports of the histories u_i are bounded, and the axiom $\mathbb{L}$.T.2 would be applicable. Not wishing to foreclose the infinite integrals for economic valuations, i.e., not to judge the termination of a production plan, the axiom $\mathbb{L}$.T.2 is chosen to serve this purpose.

2.2.4 Axioms on Boundedness of Efficient Input Histories

There are three cases of interest:

(a) $\mathbb{L}(u)$ not empty and u_i summable on $[0,+\infty)$ for all $i \in \{1,2, \ldots, m\}$

$(\tilde{\mathbb{E}})$ (a.1) $\mathbb{E}(u)$ is bounded

$(\tilde{\mathbb{E}}.S)$ (a.2) $\mathbb{E}(u)$ is totally bounded,

(b) $\mathbb{L}(u) \varepsilon 2^{(L_\infty)^n_+}$ not empty, $u \varepsilon (L_\infty)^m_+$

$(\mathbb{E})$ (b.1) $\mathbb{E}(u)$ is bounded

$(\mathbb{E}.S)$ (b.2) $\mathbb{E}(u)$ is totally bounded,

(c) $\mathbb{L}_1(u) \varepsilon 2^{(L_1)^n_+}$ not empty, $u \varepsilon (L_1)^m_+$

$(\mathbb{E}_1)$ (c.1) $\mathbb{E}_1(u)$ is bounded

$(\mathbb{E}_1.S)$ (c.2) $\mathbb{E}_1(u)$ is totally bounded.

In the third case the production correspondence is taken as $u \varepsilon (L_1)^m_+ \to \mathbb{L}_1(u) \varepsilon 2^{(L_1)^n_+}$, in products of L_1 spaces under the same axioms as used for $u \varepsilon (L_\infty)^m_+ \to \mathbb{L}(u) \varepsilon 2^{(L_\infty)^n_+}$. Then all output histories are taken as summable with norms

$$||x_i|| = \int_0^\infty x_i(t)d\nu_i(t) \ , \ i \varepsilon \{1,2, \ldots, n\}$$

$$||u_i|| = \int_0^\infty u_i(t)d\mu_i(t) \ , \ i \varepsilon \{1,2, \ldots, m\}$$

$$||x|| = \operatorname*{Max}_i ||x_i|| \ , \ ||u|| = \operatorname*{Max}_i ||u_i|| \ .$$

Clearly $\mathbb{E}.S \Rightarrow \tilde{\mathbb{E}}.S$, $\mathbb{E} \Rightarrow \tilde{\mathbb{E}}$, $\tilde{\mathbb{E}}.S \Rightarrow \tilde{\mathbb{E}}$, $\mathbb{E}.S \Rightarrow \mathbb{E}$, $\mathbb{E}_1.S \Rightarrow \mathbb{E}_1$. The stronger axioms $\mathbb{E}.S$, $\tilde{\mathbb{E}}.S$, $\mathbb{E}_1.S$ are equivalent to the corresponding efficient subsets $\mathbb{E}(u)$, $\tilde{\mathbb{E}}(u)$, $\mathbb{E}_1(u)$ being relatively compact. For the weaker axioms, the same efficient subsets are weak relatively compact under a weak* topology. Here one has to take property $\mathbb{L}.5$ as holding in the weak* topology for $(L_\infty)^n_+$ designated $\mathbb{L}.5^*$.

Proposition (2.2.4-1):

$\tilde{\mathbb{L}}(u) \subset \left(\text{CLOSURE } \tilde{\mathbb{E}}(u) + (L_\infty)^n_+\right)$ for $\mathbb{L}(u)$ not empty and u_i summable on $[0,+\infty)$. For case (a.1) the weak* topology is used with $\mathbb{L}.5^*$.

Let $y \varepsilon \tilde{\mathbb{L}}(u)$ be arbitrarily chosen, and define

$$\tilde{D}(y) := \left\{x \; \varepsilon \; (L_\infty)^n_+ : x \leqq y\right\} .$$

$$\tilde{\mathbb{K}}(u) := \{\lambda x : x \; \varepsilon \; \text{CLOSURE } \tilde{\mathbb{E}}(u) \, , \; \lambda \; \varepsilon \; [0,+\infty)\} .$$

The intersection $\mathbb{L}(u) \cap \tilde{D}(y)$ is bounded and closed. Either (1) $y \; \varepsilon \; \tilde{\mathbb{K}}(u)$ or (2) $y \notin \tilde{\mathbb{K}}(u)$. If $y \; \varepsilon \; \tilde{\mathbb{K}}(u)$, the ray $\{\lambda y : \lambda \; \varepsilon \; R_+\}$ intersects CLOSURE $\tilde{\mathbb{E}}(u)$ at a vector x , with $y = x + (y-x)$. Now $x \; \varepsilon$ CLOSURE $\tilde{\mathbb{E}}(u)$ while $(y - x) \; \varepsilon \; (\tilde{L}_\infty)^n_+$, and $y \; \varepsilon \left(\text{CLOSURE } \tilde{\mathbb{E}}(u) + (\tilde{L}_\infty)^n_+\right)$. In case (2), clearly $\tilde{\mathbb{K}}(u) \cap \tilde{D}(y) \cap \mathbb{L}(u)$ is not empty. Let z^* denote the input vector yielding

$$\text{Inf} \left\{ \int_0^\infty \left(\sum_1^n z_i(t) d\mu_i(t) \right) : z \; \varepsilon \; \tilde{\mathbb{K}}(u) \cap \tilde{D}(y) \cap \mathbb{L}(u) \right\} .$$

Since the constraint set for z is closed, $z^* \; \varepsilon$ CLOSURE $\tilde{\mathbb{E}}(u)$ and $(y - z^*) \; \varepsilon \; (\tilde{L}_\infty)^n_+$. Hence $y = z^* + (y - z^*)$ belongs to $\left(\text{CLOSURE } \tilde{\mathbb{E}}(u) + (\tilde{L}_\infty)^n_+\right)$.

For cases (b) and (c) the corresponding propositions are:

Proposition (2.2.4-2):

For $u \; \varepsilon \; (L_\infty)^m_+ \rightarrow \mathbb{L}(u) \; \varepsilon \; 2^{(L_\infty)^n_+}$, $\mathbb{L}(u) \neq \emptyset$, $\mathbb{L}(u) \subset \left(\text{CLOSURE } \mathbb{E}(u) + (L_\infty)^n_+\right)$.

Proof of Proposition (2.2.4-2):

If $u = 0$, then $\mathbb{L}(u) = (L_\infty)^n_+$, $\mathbb{E}(u) = 0$ and the proposition holds. Therefore let $y \in \mathbb{L}(u)$, $\mathbb{L}(u) \neq \emptyset$ $u \geq 0$, and define $D(y) := \left\{x \in (L_\infty)^n_+ \mid x \leqq y\right\}$. From the construction of the proof of Proposition (2.2.3-3) it is clear that the intersection (CLOSURE $\mathbb{E}(u) \cap D(y)$) is nonempty. It is compact since CLOSURE $\mathbb{E}(u)$ is compact under $\mathbb{E}$.S and $D(y)$ is closed, hence there is input vector $z^* \in$ (CLOSURE $\mathbb{E}(u) \cap D(y)$ such that $||z^*|| =$ Min $\{||z|| : z \in (\text{CLOSURE } \mathbb{E}(u) \cap D(y))\}$. Consequently, $y = z^* + (y - z^*)$ where $z^* \in$ CLOSURE $\mathbb{E}(u)$ and $(y - z^*) \in (L_\infty)^n_+$. A similar argument applies for the axiom $\mathbb{E}$ if the weak* topology is used under $\mathbb{L}.5^*$.

Proposition (2.2.4-3):

For $u \in (L_1)^m_+ \to \mathbb{L}_1(u) \in 2^{(L_1)^n_+}$, $\mathbb{L}(u) \neq \emptyset$, $\mathbb{L}(u) \subset$ CLOSURE $\mathbb{E}_1(u) + (L_1)^n_+$.

If the free-disposal-of-inputs-axiom $\mathbb{L}$.3SS is invoked, the inclusion signs $\subset$ of these last two propositions may be changed to equal signs = .

In considering the laws of return (see Chapter 3) several forms of the laws may be stated depending upon whether $u \in (L_\infty)^m_+$, $u \in (\tilde{L}_\infty)^m_+$ or $u \in (L_1)^m_+$. In the norm topologies of $(L_\infty)^m_+$, $(L_\infty)^n_+$, $\tilde{\mathbb{E}}$.S is a weaker axiom than $\mathbb{E}$.S, suitable when only summable output histories are of interest. However, the law is accordingly a weak statement in that it restricts consideration to summable input histories, as compared to those permitted for the law with $\mathbb{E}$.S applied. A similar relationship exists for the axioms $\tilde{\mathbb{E}}$ and $\mathbb{E}$ under weak* topologies. Finally under a topology for summable histories which norms them by the total amount inputed or outputed, i.e., the norm topology for L_1 spaces, still other expressions of the laws of return are obtained.

2.2.5 Weak Topology for the Input Space

Let p_i $(i = 1,2, \ldots, n)$ denote a real valued function defined on R_+ corresponding to the input history x_i, which is summable in absolute value, i.e.,

$$\int_0^\infty |p_i(t)|d\mu_i(t) < +\infty \ , \ i \in \{1,2, \ldots, n\}$$

with respect to the sigma-finite measure space $\left(R_+, \sum_i, \mu_i\right)$. Moreover, p_i is taken as a function element of a space $L_1\left(R_+, \sum_i, \mu_i\right)$ of μ_i-Equivalence classes normed by

$$(2.2.5\text{-}1) \qquad ||p_i|| = \int_0^\infty |p_i(t)|d\mu_i(t) \ .$$

The vector $p = (p_1, p_2, \ldots, p_n)$ is then taken as an element of the product space

$$\mathop{X}_{i=1}^{n} L_1\left(R_+, \sum_i, \mu_i\right) := (L_1)^n$$

of absolute value summable equivalence classes relative to the measure $\mu = (\mu_1, \ldots, \mu_n)$. The norm of p is taken as

$$(2.2.5\text{-}2) \qquad ||p|| = \operatorname*{Max}_i ||p_i|| \ , \ i \in \{1,2, \ldots, n\} \ .$$

The space $L_\infty\left(R_+, \sum_i, \mu_i\right)$ of equivalence classes for the i^{th} exogenous input is isometric isomorphic to the dual space $L_1^*\left(R_+, \sum_i, \mu_i\right)$. The weak* topology for $L_\infty\left(R_+, \sum_i, \mu_i\right) := (L_\infty)_i$ is a base of neighborhoods χ_i defined at 0 (the null input history equivalence class) consisting of open sets

$$\text{(2.2.5-3)} \qquad \left\{ x_i \in (L_\infty)_i : \sup_{\alpha=1,2,\ldots,k} \int_0^\infty |p_i^\alpha(t)x_i(t)d\mu_i(t)| < \epsilon \right\}$$

for $k \in \{1,2, \ldots\}$ where $\left\{p_i^1, p_i^2, \ldots, p_i^k\right\}$ is a finite collection of elements from $L_1\left(R_+, \sum_i, \mu_i\right)$. Then the weak* topology for

$$(L_\infty)^n := \underset{i=1}{\overset{n}{\mathrm{X}}} L_\infty\left(R_+, \sum_i, \mu_i\right)$$

has a neighborhood base at 0 given by

$$\text{(2.2.5-4)} \qquad \chi = \underset{i=1}{\overset{n}{\mathrm{X}}} \chi_i \, .$$

The norm of x_i in this weak* topology may be taken

$$\text{(2.2.5-5)} \qquad ||x_i|| = \sup_{||p_i|| \leqq 1} \left(\int_0^\infty |p_i(t)x_i(t)d\mu_i(t)| \right)$$

in place of (1-1) with $||x||$ defined by (1-6).

This topology is the weakest topology for which the linear functional

$$\text{(2.2.5-6)} \qquad \langle x_i, p_i \rangle = \int_0^\infty p_i(t)x_i(t)d\mu_i(t)$$

is continuous.

One may regard the function p_i as a price history for the input history x_i, summable in absolute value. For the purposes of economic analysis one usually considers only nonnegative prices for exogenous inputs. However this is no impediment in the use of the weak*

topology for one's purposes, since we shall restrict consideration to the subset of nonnegative functions p_i .

By the use of this weak topology one may use the weaker axioms $\tilde{E}(u)$, $(\mathbb{E}(u))$ for the efficient subset and still retain compactness for the closure of $\tilde{\mathbb{E}}(u)$, $(\mathbb{E}(u))$.

2.3 Freedom of Axioms from Contradiction

The axioms for the dynamic production correspondence are not contradictory if there exists a dynamic correspondence $x \in (L_\infty)_+^n \to \mathbb{K}(x) \in 2^{(L_\infty)_+^m}$ which satisfies them. Examples of the correspondence $x \to \mathbb{K}(x)$ for this purpose will be shown here which are dynamic generalizations of the familiar C.E.S. and Leontief steady state production functions. When used for econometric study these examples will afford a basis for explicit consideration of the dynamic structure of production.

Proposition (2.3-1):

The axiom system $S_1 := \{\mathbb{P}.1, \mathbb{P}.2, \mathbb{P}.3, \mathbb{P}.4.1, \mathbb{P}.4.2, \mathbb{P}.5, \mathbb{P}.6S, \mathbb{E}, \mathbb{P}.T.1, \mathbb{P}.T.2, \mathbb{L}.T.1, \mathbb{L}.T.2\}$ is free of contradiction.

Let $I_\nu := [(\nu - 1), \nu)$, $\nu = \{1,2, \ldots\}$ be a partition of the domain $[0,+\infty)$ for t , by some suitable unit of time. Define $Y := \left\{x \in (L_\infty)_+^n : x_j(t) = x_j(\nu) , x_j(\nu) \in R_+ , t \in I_\nu , \nu = 1,2, \ldots , j \in \{1,2, \ldots, n\}\right\}$ and consider the correspondences given by:

$$\mathbb{K}(x) := \left\{u \in (L_\infty)_+^m : u_i = \theta_i w_i , \theta_i \in [0,1] , w_i \in (L_\infty)_+ , i \in \{1,2, \ldots, m\}\right\} \tag{2.3-1}$$

$$w_i := \begin{cases} 0 \ \varepsilon \ (L_\infty)_+ & \text{if } \ x \notin Y \\ w_{i\nu} \ \varepsilon \ R_+ & \text{for } \ x \ \varepsilon \ Y \ \text{ and } \ t \ \varepsilon \ I_\nu \ , \ \nu = 1,2, \ \dots \end{cases}$$

(2.3-2)

$$w_{i\nu} := \begin{cases} 0 \quad \text{for} \quad 1 \leqq \nu \leqq \left(\nu_i^o - 1\right) \ , \ \nu_i^o > 1 \ , \\ \sum_{j=1}^{n} \beta_{j\nu}^{(i)} \sum_{\tau=\left(\nu+1-\nu_i^o\right)}^{\nu} \alpha_j^{(i)}(\nu+1-\tau)x_j(\tau) \ , \ \nu \geqq \nu_i^o \ . \end{cases}$$

The technical coefficients $\alpha_j^i(\sigma)$, $\sigma \ \varepsilon \left\{1,2, \ \dots, \ \nu_i^o\right\}$, are nonnegative and bounded for $i \ \varepsilon \ \{1,2, \ \dots, \ m\}$ and $j \ \varepsilon \ \{1,2, \ \dots, \ n\}$, and satisfy

(2.3-3) Either $\alpha_j^{(i)}(\sigma) > 0$ or $\alpha_j^{(i)}(\sigma) = 0 \ \forall \sigma \ \varepsilon \left\{1,2, \ \dots, \ \nu_i^o\right\}$.

(2.3-4) $\sum_{j=1}^{n} \alpha_j^{(i)}(\sigma) > 0 \ \forall \sigma \ \varepsilon \left\{1,2, \ \dots, \ \nu_i^o\right\}$ and each $i \ \varepsilon \ \{1,2, \ \dots, \ m\}$.

(2.3-5) $\sum_{i=1}^{m} \alpha_j^{(i)}(\sigma) > 0 \ \forall \sigma \ \varepsilon \left\{1,2, \ \dots, \ \nu_i^o\right\}$ and each $j \ \varepsilon \ \{1,2, \ \dots, \ n\}$.

For each $j \ \varepsilon \ \{1,2, \ \dots, \ n\}$, the functions $\beta_j^{(i)} \ \varepsilon \ (L_\infty)_+$, $\beta_j^{(i)}(t) = \beta_{j\nu}^{(i)} \ \varepsilon \ R_+$, $t \ \varepsilon \ I_\nu$, $\nu = 1,2, \ \dots$ are nonnegative and bounded, but not summable otherwise only summable output histories u_i are possible, satisfying

(2.3-6) $\beta_j^{(i)}(t) = 0$ for $t \ \varepsilon \left(\bar{t}_{x_j}, +\infty\right)$, $j \ \varepsilon \ \{1,2, \ \dots, \ n\}$.

(2.3-7) $\beta^{(i)} := \left(\beta_1^{(i)}, \beta_2^{(i)}, \ \dots, \ \beta_n^{(i)}\right) \geq 0$, $i \ \varepsilon \ \{1,2, \ \dots, \ m\}$.

Inputs prior to zero time are taken zero.

Condition (2.3-3) is merely a simplifying assumption of consistency, i.e., if an input history is involved in the production of an output rate $w_{i\nu}$, $\nu \geqq \nu^o$, it is involved throughout the period I_ν and the preceding periods I_σ , $\sigma = 2,3, \ldots \left(\nu_i^o - 1\right)$, while conditions (2.3-4) and (2.3-5) state respectively that each output uses at least one input and each input is used in at least one output.

This representation of a dynamic correspondence gives explicit recognition of the discrete nature of observed data as an approximation of the underlying process. For certain regularity of the underlying processes, one may approximate a dynamic production correspondence to desired accuracy by choice of the unit of time. For the correspondence so modeled, a positive period of time of length ν_i^o is required to charge-up the production system for the i^{th} output, during which none of the output is obtained. Thereafter, the i^{th} output at any time t is not affected by inputs applied ν_i^o or more units of time earlier. Exogenous inputs to inventory are not explicitly allowed for, and the input histories x_j refer to actual usage in production. The length of the period to charge-up the system for the i^{th} output may be taken as a positive integer without serious loss of generality. Step function input rate histories generate step function output rate histories.

Taking the properties of S_1 in turn, consider $\mathbb{P}.1$ first by supposing that x is (essentially) null for $x \to \mathbb{K}(x)$. If $x \notin Y$, $\mathbb{K}(x) = \{0\}$. If $x \in Y$, clearly the functions w_i are zero for all $t \in [0,+\infty)$, and $i \in \{1,2, \ldots, m\}$. Hence by (2.3-1), $\mathbb{K}(0) = \{0\}$, i.e., the vector of null output histories is the only possibility. Thus property $\mathbb{P}.1$ holds for $x \to \mathbb{K}(x)$.

For bounded input histories x_j, it is clear from (2.3-2) that the functions w_i are bounded, and by (2.3-1) it follows that $\mathbb{K}(x)$ is bounded.

In the case of property $\mathbb{P}.3$, let $x' = \lambda x$ where $\lambda \in [1,+\infty)$. If $x \notin Y$, $x' \notin Y$ and $w_i(x') = w_i(x) = 0 \in (L_\infty)_+$, so that by (2.3-1) $\mathbb{K}(x') = \mathbb{K}(x)$ and property $\mathbb{P}.3$ holds. If $x \in Y$, $x' \in Y$ and, since $w_{i\nu}$ is homogeneous in x, it follows that $w_{i\nu}(x') = \lambda w_{i\nu}(x)$, $\nu = 1,2, \ldots$ and $\mathbb{K}(x) \subset \mathbb{K}(x')$. Hence property $\mathbb{P}.3$ is fulfilled.

It is convenient here to note that $\mathbb{K}(x) \neq \{0\}$ on $(L_\infty)^n_+$, since the coefficients $\alpha^i_j(\sigma)$, $\sigma = 1,2, \ldots, \nu^o_i$, and the coefficient functions $\beta^i_j(t)$, $t \in [0,+\infty)$ satisfy (2.3-3), (2.3-4), (2.3-5) and (2.3-6), (2.3-7) respectively. Property $\mathbb{P}.4.1$ is seen to hold for $x >> 0$, and property $\mathbb{P}.4.2$ clearly holds for $\lambda_\theta = \theta$.

Regarding property $\mathbb{P}.5$, let $\{x^\alpha\} \to x^o$, $u^\alpha \in \mathbb{K}(x^\alpha)$ for all $\alpha = 1,2, \ldots$, with $\{u^\alpha\} \to u^o$. Either $x^\alpha \in Y$ for almost all α or $x^\alpha \notin Y$ for almost all α. In the latter case, $u^o = 0$ and $u^o \in \mathbb{K}(x^o)$. In the former case $u^\alpha = \theta^\alpha w_i(x^\alpha)$ for $\theta^\alpha \in [0,1]$, $\alpha = 1,2, \ldots$. Clearly there exists a subsequence $\{\alpha_k\}$ such that $\left\{\theta^{\alpha_k}\right\} \to \theta^o$, $\theta^o \in [0,1]$. Further $\left\{u^{\alpha_k}\right\} \to u^o$ and $\left\{w_i\left(x^{\alpha_k}\right)\right\} \to w_i(x^o)$. Hence $u^o = \theta^o \cdot w_i(x^o)$, $\theta^o \in [0,1]$. Thus $u^o \in \mathbb{K}(x^o)$, and property $\mathbb{P}.5$ holds.

To complete the sequence of $\mathbb{P}$ properties, it is obvious from the very definition (2.3-1) of $x \to \mathbb{K}(x)$ that property $\mathbb{P}.6S$ holds.

Property $\mathbb{E}$ is next for consideration. Let $u \in (L_\infty)^m_+$ with $\mathbb{L}(u)$ not empty. For any such vector of output functions, $||u_i|| \leqq B$, $B \in R_{++}$ for all $i \in \{1,2, \ldots, m\}$. Clearly, if $x \in \mathbb{E}(u)$, then

$u_i(t) = w_i(t \mid x)$, $t \in [0,+\infty)$, $i \in \{1, \ldots, m\}$. Thus for $x \in \mathbb{E}(u)$ it is necessary that $||w_i(x)|| \leqq B$, for all $i \in \{1,2, \ldots, m\}$. Now assume that there exists a sequence $\{x^\alpha\} \subset \mathbb{E}(u)$ such that $\{||x^\alpha||\} \to +\infty$. Then, it would follow from (2.3-2) $\cdots$ (2.3-7) that $\left\{||w_i(x^\alpha)||\right\} \to +\infty$ for some $i \in \{1,2, \ldots, m\}$, contradicting $||w_i|| < B$ for all $i \in \{1,2, \ldots, m\}$. Hence $\mathbb{E}(u)$ is bounded and property $\mathbb{E}$ holds.

Concerning property $\mathbb{P}$.T.1 on the initiation of output (see §2.1.3) stating that outputs can occur only after an interval has occurred since initiation of inputs, this property is assured by the definitions of $w_i(t)$, i.e., (2.3-2). Concerning property $\mathbb{P}$.T.2 stating that all outputs cease when all inputs cease, condition (2.3-6) assures this.

It is clear from conditions (2.3-3) $\cdots$ (2.3-7) that if w_i is summable for all $i \in \{1,2, \ldots, m\}$, and $x \in (L_\infty)^n_+$ *satisfies (2.3-2)* under $t \in \left(\nu^o_i,+\infty\right)$ for w_i , then x_i may be summable in each component. Thus property $\mathbb{L}$.T.1 holds. Finally concerning $\mathbb{L}$.T.2, it holds trivially because if $\bar{t}_u < +\infty$ (see §2.2.2) it follows from (2.3-1) $\cdots$ (2.3-7) that if $x \in \mathbb{L}(u)$ then $x_j(t) = 0$ for $t > \bar{t}_u$ and $j \in \{1,2, \ldots, n\}$ is possible.

It is to be noted that properties $\mathbb{P}$.2S and $\mathbb{P}$.6SS cannot be used together, because under $\mathbb{P}$.6SS, $\mathbb{P}(x)$ may not be totally bounded. To see this, consider

$$\mathbb{P}(x) = \{u \in (L_\infty)_+ : 0 \leqq u \leqq x\} , \; x \in (L_\infty)_+ .$$

The example (2.1.1-1) of Section 2.1.1 may be applied by choosing $x(t) = 1$, $t \in [0,+\infty)$ and taking the sequence of functions belonging to $\mathbb{P}(x)$ as

$$u^{(i)}(t) = \begin{cases} 1 & \text{if} \quad t \in \left[i, i + \frac{1}{2}\right] \\ 0 & \text{otherwise.} \end{cases}$$

Thus property $\mathbb{P}.6SS$ opposes property $\mathbb{P}.2S$. The crux of the matter here is that $\mathbb{P}.6SS$ permits free decrease of the time distribution of output histories u belonging to $\mathbb{P}(x)$. However, if the output set $\mathbb{P}(x)$ is modified to

$$\mathbb{P}(x) = \{u \in (L_\infty)_+ : u = \theta x \ , \ \theta \in [0,1]\} \ , \ x \in (L_\infty)_+$$

only scaling of a given time distribution is permitted, and output sets of this type satisfy $\mathbb{P}.6S$ along with $\mathbb{P}.2S$.

In this connection it is useful to establish the following general result.

<u>Proposition (2.3-2)</u>:

Properties $\mathbb{P}.2S$ and $\mathbb{P}.6S$ are compatible.

<u>Proof</u>:

Let $\{u^\alpha\}$ denote an arbitrary infinite sequence of vectors of output functions belonging to an output set $\mathbb{P}(x)$ satisfying $\mathbb{P}.2S$. This set is totally bounded if and only if each infinite sequence $\{u^\alpha\} \subset \mathbb{P}(x)$ has a Cauchy subsequence, since $(L_\infty)^m$ is a metric space. Now for $\mathbb{P}.6S$, $u_i^\alpha = \theta_i^\alpha w_i^\alpha$, $i \in \{1, \ldots, m\}$, where $w^\alpha = \left(w_1^\alpha, \ldots, w_m^\alpha\right)$ is a sequence of output vectors of $\mathbb{P}(x)$ and $\theta_i^\alpha \in [0,1]$ for all $i \in \{1, \ldots, m\}$ and $\alpha = 1,2, \ldots$. The sequences $\left\{\theta_i^\alpha\right\}$ have convergent subsequences $\left\{\theta_i^{\alpha_k}\right\} \to \theta_i^o$ common for $i \in \{1,2, \ldots, m\}$.

Hence, consider the subsequences $\left\{\theta_i^{\alpha_\ell} w_i^{\alpha_\ell}\right\}$ of $\{u^\alpha\}$. Now, since $\left\{w^{\alpha_\ell} = \left(w_1^{\alpha_\ell}, w_2^{\alpha_\ell}, \ldots, w_m^{\alpha_\ell}\right)\right\} \subset \mathbb{P}(x)$ for $\mathbb{P}.2S$ holdings, there exists a further subsequence $\left\{w^{\alpha_k} = \left(w_1^{\alpha_k}, \ldots, w_m^{\alpha_k}\right)\right\} \to w^o = \left(w_1^o, \ldots, w_m^o\right)$. Hence, it remains only to consider the subsequence $\left\{\theta_i^{\alpha_k} w_i^{\alpha_k}\right\}$ of $\left\{u_i^\alpha\right\}$ showing that $\left\{\theta_i^{\alpha_k} w_i^{\alpha_k}\right\} \to \theta_i^o w_i^o$. For this purpose, write

$$
\begin{aligned}
||\theta_i^{\alpha_k} w_i^{\alpha_k} - \theta_i^o w_i^o|| &= ||\theta_i^o\left(w_i^{\alpha_k} - w_i^o\right) + \left(\theta_i^{\alpha_k} - \theta_i^o\right)w_i^o + \left(\theta_i^{\alpha_k} - \theta_i^o\right)\left(w_i^{\alpha_k} - w_i^o\right)|| \\
&\leqq \theta_i^o \cdot ||w_i^{\alpha_k} - w_i^o|| + |\theta_i^{\alpha_k} - \theta_i^o| \cdot \left(||w_i^o|| + ||w_i^{\alpha_k} - w_i^o||\right)
\end{aligned}
$$

and, as $\alpha_k \to \infty$, $\theta_i^{\alpha_k} w_i^{\alpha_k} \to \theta_i^o w_i^o$. Thus $\mathbb{P}(x)$ is totally bounded under $\mathbb{P}.6S$ when $\mathbb{P}.2S$ holds.

<u>Sub-Proposition (2.3-2)</u>:

Properties $\mathbb{P}.2S$ and $\mathbb{P}.6$ are compatible.

<u>Sub-Proposition (2.3-1)</u>:

Property $\mathbb{P}.6$ may be used for property $\mathbb{P}.6S$ in the axiom system S_1 of Proposition (2.3-1) and $\tilde{\mathbb{E}}$ may be substituted for $\mathbb{E}$ in S_1.

Clearly, if $\mathbb{P}.6S$ is satisfied then property $\mathbb{P}.6$ holds merely by taking $\theta_i = \theta$ for $i \in \{1,2, \ldots, m\}$. Further, since $\mathbb{E}$ implies $\tilde{\mathbb{E}}$, the latter is likewise satisfied for the correspondence $x \to \mathbb{K}(x)$.

So far we have not considered the stronger property $\mathbb{P}.2S$ that the output sets $\mathbb{K}(x)$ are totally bounded. The following proposition shows this property to be fulfilled by the correspondence $x \to \mathbb{K}(x)$ along with those previously considered.

Proposition (2.3-3):

The axiom system S_2 : {$\mathbb{P}$.1, $\mathbb{P}$.2S, $\mathbb{P}$.3, $\mathbb{P}$.4.1, $\mathbb{P}$.4.2, $\mathbb{P}$.5, $\mathbb{P}$.6S, $\mathbb{E}$, $\mathbb{P}$.T.1, $\mathbb{P}$.T.2, $\mathbb{L}$.T.1, $\mathbb{L}$.T.2} is free of contradiction.

Reconsider the correspondence (2.3-1) ··· (2.3-7). We need only consider property $\mathbb{P}$.2S. This property holds because a single vector $w = (w_1, w_2, \ldots, w_m)$ of output histories is obtained from (2.3-2) for each $x \in (L_\infty)^n_+$, with all vectors of output histories of $\mathbb{K}(x)$ obtained from w by scaling separately the components of w_i $(i = 1,2, \ldots, m)$ so as to preserve the time distribution of w_i. Since $(L_\infty)^m$ is a Banach space with metric $\rho(x,y) = ||x - y||$, $\mathbb{K}(x)$ is totally bounded in the metric if and only if each infinite sequence $\{u^\alpha\} \subset \mathbb{K}(x)$ has a Cauchy subsequence. Now by the definition of $\mathbb{K}(x)$, $u_i^\alpha = \theta_i^\alpha w_i$. Since $\theta_i^\alpha \subset [0,1]$, this sequence has a convergent subsequence $\left\{\theta^{\alpha_k}\right\} \to \theta_i^o$. Hence $\left\{\theta_i^{\alpha_k} w_i\right\} \to \theta_i^o w_i$ for each $i \in \{1,2, \ldots, m\}$. Thus $\{u^\alpha\} \to u^o$, where $u^o = \left(\theta_1^o w_1, \ldots, \theta_m^o w_m\right)$, and $\mathbb{K}(x)$ is totally bounded (relatively compact).

Sub-Proposition (2.3-3):

Property $\mathbb{P}$.6 may be substituted for property $\mathbb{P}$.6S and $\tilde{\mathbb{E}}$ for $\mathbb{E}$ in the axiom system S_2 of Proposition (2.3-3).

The definitions (2.3-1) ··· (2.3-7) used for $x \to \mathbb{K}(x)$ does not permit strong ($\mathbb{P}$.3S) or superstrong ($\mathbb{P}$.3SS) disposal of inputs. For any input vector $x \in (L_\infty)^n_+$ a time distribution of component histories u_i is determined, and by independent scaling of the components of x these time distributions are not preserved by (2.3-2).

So far we have not considered any axiom system containing $\mathbb{E}$.S. For this purpose let $U := \left\{u \in (L_\infty)_+^m : u_i(t) = u_i(\nu) \ , \ u_i(\nu) \in R_+ \ , \ t \in I_\nu \ , \ \nu = 1,2, \ldots ; \ i \in \{1,2, \ldots, m\}\right\}$ and consider the correspondence $u \in (L_\infty)_+^m \to \mathcal{M}(u) \in 2^{(L_\infty)_+^n}$ given by

$$(2.3\text{-}9) \quad \mathcal{M}(u) := \begin{cases} \left\{x \in (L_\infty)_+^n : x_j \geq y_j \ , y_j \in (L_\infty)_+ \ , \ j \in \{1,2, \ldots, n\}\right\} \text{ if } u \in U \\ \text{empty if } u \in U \ , \ u_i(t) > 0 \text{ for } t \in \left[0, \left(\nu_i^o - 1\right)\right), \ \nu_i^o > 1 \\ \text{empty if } u \notin U \ , \ u \geq 0 \\ (L_\infty)_+^n \text{ if } u = 0 \ . \end{cases}$$

$$(2.3\text{-}10) \quad y_j(t) = y_{j\nu} \in R_+ \text{ for } u \in U \ , \ t \in I_\nu \ , \ \nu = 1,2, \ldots \ , \ j \in \{1, \ldots, m\} \ .$$

$$(2.3\text{-}11) \quad y_{j\nu} = \sum_{i=1}^{m} b_{i\nu}^{(j)} \sum_{\tau=\nu}^{\left(\nu+\nu_i^o-1\right)} c_i^{(j)}(\tau + 1 - \nu) u_i(\tau) \ , \ (\nu = 1,2, \ldots) \ .$$

The technical coefficients $c_i^{(j)}(\sigma)$, $\sigma \in \left\{1,2, \ldots, \nu_i^o\right\}$, are non-negative and bounded for $i \in \{1,2, \ldots, m\}$, $j \in \{1,2, \ldots, n\}$, and satisfy

$$(2.3\text{-}12) \quad \text{Either } c_i^{(j)}(\sigma) > 0 \text{ or } c_i^{(j)}(\sigma) = 0 \ \forall \sigma \in \left\{1,2, \ldots, \nu_i^o\right\} .$$

$$(2.3\text{-}13) \quad \sum_{i=1}^{m} c_i^{(j)}(\sigma) > 0 \ \forall \sigma \in \left\{1,2, \ldots, \nu_i^o\right\} \text{ and each } j \in \{1,2, \ldots, n\} \ .$$

$$(2.3\text{-}14) \quad \sum_{j=1}^{n} c_i^{(j)}(\sigma) > 0 \ \forall \sigma \in \left\{1,2, \ldots, \nu_i^o\right\} \text{ and each } i \in \{1,2, \ldots, m\} \ .$$

For all $i \in \{1,2, \ldots, m\}$ the functions $b_i^{(j)} \in (L_\infty)_+ : b_i^{(j)}(t) = b_{i\nu}^{(j)}$, $t \in I_\nu$, $\nu = 1,2, \ldots$ are nonnegative and bounded, and not summable otherwise only summable input histories may be applied, satisfying

$$(2.3\text{-}15)\quad b^{(j)} := \left(b_1^{(j)}, b_2^{(j)}, \ldots, b_m^{(j)}\right) \geq 0 \ , \ j \in \{1,2, \ldots, n\} \ .$$

Similar to the conditions (2.3-4), (2.3-5), the conditions (2.3-13), (2.3-14) state respectively that each output uses at least one input and each input is used in at least one output, and like (2.3-3), condition (2.3-12) is merely an assumption of consistency.

One observes immediately that property $\mathbb{E}$.S holds for the correspondence $u \to \mathcal{M}(u)$, since $\mathbb{E}(u)$ consists only of the vector $y = (y_1, y_2, \ldots, y_n)$ of input histories. See definition (2.2.3-1) for efficiency.

Now consider the axiom systems $S_1' := \{\mathbb{L}.1,\ \mathbb{L}.2,\ \mathbb{L}.3SS,\ \mathbb{L}.4.1,\ \mathbb{L}.4.2,\ \mathbb{L}.5,\ \mathbb{L}.6S,\ \mathbb{E}.S,\ \mathbb{P}.T.1,\ \mathbb{P}.T.2,\ \mathbb{L}.T.1,\ \mathbb{L}.T.2\}$ inverse to S with $\mathbb{E}$ replaced by $\mathbb{E}$.S and $\mathbb{L}$.3SS replacing $\mathbb{L}$.3.

Property $\mathbb{L}$.1 is satisfied by the definition (2.3-9). In regard to property $\mathbb{L}$.2 we need only show that the set $\mathcal{M}^{-1}(x)$ of the inverse correspondence $x \in (L_\infty)_+^n \to \mathcal{M}^{-1}(x) \in 2^{(L_\infty)_+^m}$ is bounded, where

$$\mathcal{M}^{-1}(x) = \left\{u \in (L_\infty)_+^m : x \in \mathcal{M}(u)\right\} \ .$$

$\mathcal{M}^{-1}(x)$ is bounded for $||x|| < +\infty$, because suppose $\{||u^\alpha||\} \to +\infty$ with $u^\alpha \in \mathcal{M}^{-1}(x)$ for all $\alpha = 1,2, \ldots$. Then for some $j \in \{1,2, \ldots, n\}$ the relations $y_j \leqq x_j$ are violated because the

conditions (2.3-12) $\cdots$ (2.3-15) guarantee that $\left\{\|y_j^\alpha\|\right\} \to +\infty$. Hence $\mathcal{M}^{-1}(x)$ is bounded.

Concerning $\mathbb{L}$.3SS, it holds by the very definition of $\mathcal{M}(u)$. Property $\mathbb{L}$.4.1 evidently holds, and $\mathbb{L}$.4.2 is satisfied, since $y_j(t \mid \theta u) = \theta y_j(t \mid u)$, $\theta \in [0,+\infty)$.

Concerning the closure of $u \to \mathcal{M}(u)$, let $\{u^\alpha\} \to u^o$, $x^\alpha \in \mathcal{M}(u^\alpha)$ for $\alpha = 1,2, \ldots$, with $\{x^\alpha\} \to x^o$. Now $u^\alpha \in U$ for almost all α with $u_i^\alpha(t) \nleq 0$ for $t \in \left[0, \left(\nu_i^o - 1\right)\right)$, otherwise $\mathcal{M}(u^\alpha)$ is empty and $x^\alpha \notin \mathcal{M}(u^\alpha)$. Then $x_j^\alpha(t) \geqq y_j^\alpha(t \mid u^\alpha)$, $\alpha = 1,2, \ldots$, $t \in [0,+\infty)$, $j \in \{1,2, \ldots, n\}$ with $\left\{x_j^\alpha(t)\right\} \to x_j^o(t)$ and $\left\{y_j^\alpha(t \mid u^\alpha)\right\} \to y_j^o(t)$, where

$$y_j^o(t) = y_{j\nu}^o \in R_+ \text{, and } y_{j\nu}^o = \sum_{i=1}^{m} b_{i\nu}^{(j)} \sum_{\tau=\nu}^{\left(\nu+\nu_i^o-1\right)} c_i^{(j)}(\tau + 1 - \nu) u_i^o(\tau) \,,$$

$\nu = 1,2, \ldots$. Hence $x^o(t) \geqq y_j^o(t)$, $t \in [0,+\infty)$, $j \in \{1,2, \ldots, n\}$ and $x^o = \left(x_1^o, \ldots, x_n^o\right) \in \mathcal{M}(u^o)$.

Property $\mathbb{L}$.6S holds for $u \to \mathcal{M}(u)$, since if $x \in \mathcal{M}(u)$, $u^1 := (\theta_1 u_1, \theta_2 u_2, \ldots, \theta_m u_m) \geqq u$ for $\theta_i \in [1,+\infty)$, $i \in \{1, \ldots, m\}$, and

$$x_j \geqq y_j(u^1) \geqq y_j(u) \,, \; j \in \{1,2, \ldots, n\}$$

and $x \in \mathcal{M}(u)$.

Concerning property $\mathbb{P}$.T.1, it is guaranteed by (2.3-9) and $\mathbb{P}$.T.2 results from (2.3-9), (2.3-10), (2.3-11). If u is summable in each component on $[0,+\infty)$, y_j is summable on $[0,+\infty)$ for each $j \in \{1,2, \ldots, n\}$ and $x_j = y_j$ is summable for all j. Hence $\mathbb{L}$.T.1 holds. Finally, it is clear from the definitions (2.3-9), (2.3-10), (2.3-11) that $\mathbb{L}$.T.2 holds.

Thus the following proposition holds:

Proposition (2.3-4):

The axiom system S_1' , or S_1 with $\mathbb{P}.3$ replaced by $\mathbb{P}.3SS$ and $\mathbb{E}$ replaced by $\mathbb{E}.S$, is free of contradiction, and $\mathbb{E}.S$ may be replaced by $\tilde{\mathbb{E}}.S$.

It remains to consider properties $\mathbb{E}.S$ (or $\tilde{\mathbb{E}}.S$) substituting for $\mathbb{E}$ (or $\tilde{\mathbb{E}}$) in the axiom system S_2 . For this case consider the dynamic production correspondence $x \to \mathbb{K}(x)$ defined only for $x \in \left\{\lambda x^o : \lambda \in [0,+\infty) , x^o \in (L_\infty)^n_+\right\}$ by (2.3-1), (2.3-2) $\cdots$ (2.3-7). Clearly all the properties of S_2 hold. Further, since x is confined to a single vector of input histories altered only by scaling these histories by a common factor λ , for the inverse correspondence $u \in (L_\infty)^m_+ \to \mathbb{K}^{-1}(u) \in 2^{(L_\infty)^n_+}$

$$\mathbb{K}^{-1}(u) = \left\{x : x = \lambda x^o , u \in \mathbb{K}(x)\right\}$$

when not empty, and hence there is a single efficient vector for each $\mathbb{K}^{-1}(u)$ not empty, and $\mathbb{E}(u)$ is totally bounded. Hence the following proposition holds.

Proposition (2.3-5):

The axiom system S_2 with $\mathbb{E}$ (or $\tilde{\mathbb{E}}$) replaced by $\mathbb{E}.S$ (or $\tilde{\mathbb{E}}.S$) is free of contradiction.

One additional example is useful to illustrate how totally bounded efficient subsets $\mathbb{E}(u)$ may be quite generally obtained, and at the same time provide an example of a dynamic neoclassical production function.

The construction to be used is dictated by a proposition in [(Dunford and Schwartz, 1967), p. 342] that a bounded subset $K \subset L_\infty(R_+, \sum, \mu)$ is compact if and only if there exists for each $\epsilon > 0$ a partition π of R_+ into a finite number of measurable sets such that μ-Ess Sup $|f(s) - f(t)| < \epsilon$ for $s \in A$, $t \in A$, and each subset A of π.

For the representation of time, the nonnegative real line is partitioned by

$$[0,+\infty) = \left(\bigcup_1^N [(\nu - 1),\nu)\right) \cup [N,+\infty) ,$$

for some convenient unit of time. A set X of vectors of input histories is defined by

$$\begin{aligned} X := \Big\{ x \in (L_\infty)_+^n : x_j(t) = x_j(\nu) ,\ x_j(\nu) \in R_+ ,\ t \in [(\nu - 1),\nu) , \\ (\nu = 1,2, \ldots, N) ,\ x_j(t) = x_j(N + 1) \\ x_j(N + 1) \in R_+ ,\ t \in [N,+\infty) ,\ j \in \{1,2, \ldots, n\} \Big\} . \end{aligned}$$

We note that if $\{x^n\} \subset X$ is a Cauchy sequence $\{x^n\} \to x^o \in X$.

A dynamic correspondence $x \in (L_\infty)_+^n \to \mathbb{K}(x) \in 2^{(L_\infty)_+}$ is considered where

$$\mathbb{K}(x) = \{u \in (L_\infty)_+ : u = \theta \, \phi\!\!\phi(x) ,\ \theta \in [0,1]\} ,\ x \in (L_\infty)_+^n$$

and $\phi\!\!\phi(x) \in (L_\infty)_+$ is defined by:

$$\phi\!\!\phi(x) := \begin{cases} 0 \quad \text{for} \quad x \notin X \\ \phi_i \quad \text{for} \quad t \in [(i-1),i) ,\ \text{for} \quad i \in \{1,2, \ldots, N\} ,\ \text{and} \\ \qquad\qquad t \in [N,+\infty) \quad \text{for} \quad i = (N + 1) ,\ x \in X \end{cases}$$

where

$$\phi_i := \begin{cases} 0 \quad \text{for} \quad 1 \leqq i \leqq (\kappa_o - 1)\ ,\ \kappa_o > 1 \\ \\ \sum_{j=1}^{n} \beta_{ij} \sum_{\nu=(i+1-\kappa_o)}^{i} \alpha_j(i+1-\nu)x_j(\nu)\ ,\ i \in \{\kappa_o, \kappa_o+1, \ldots, N, N+1\} \end{cases}$$

and the coefficients $\alpha_j(\sigma)$ are positive and bounded for $(\sigma = 1, \ldots, \kappa_o)$. The coefficients β_{ij} are positive and bounded for $\left(1 \leqq i < \bar{t}_{x_j}\right)$, with $\beta_{ij} = 0$ for $i \geqq \bar{t}_{x_j}$, $j \in \{1,2, \ldots, n\}$.

For $x = 0$, $\phi(x) = 0$ and $\mathbb{K}(0) = \{0\}$. Also for $||x|| < +\infty$, $||\phi|| < +\infty$ and $\mathbb{K}(x)$ is bounded. Thus properties $\mathbb{P}$.1 and $\mathbb{P}$.2 hold for $x \to \mathbb{K}(x)$. Further, as defined, $\mathbb{K}(x)$ is totally bounded and $\mathbb{P}$.2S , and perforce $\mathbb{P}$.2, hold. Moreover, it is seen that $\phi(\lambda x) = \lambda\,\phi(x)$ for $\lambda \in [0,+\infty)$. Consequently property $\mathbb{P}$.3 holds. Further, if $x \in X$, $(\lambda_1 x, \lambda_2 x_2, \ldots, \lambda_n x_n) \in X$ for $\lambda_j \geqq 1$, $j \in \{1,2, \ldots, n\}$ (see definition of X), and $\phi(\lambda_1 x_1, \ldots, \lambda_n x_n) \geqq \phi(x)$. Hence property $\mathbb{P}$.3S holds. Property $\mathbb{P}$.3SS cannot hold because y need not belong to X for $y \geq x$ when $x \in X$. For $x \notin X$, $\mathbb{P}$.3S and $\mathbb{P}$.3 hold trivially.

Since the coefficients $\alpha_j(\sigma)$, $\sigma = 1,2, \ldots, \kappa_o$ are positive and $\beta_j(i) > 0$ for $\left(1 \leqq i < \bar{t}_{x_j}\right)$, property $\mathbb{P}$.4.1 is immediate. Property $\mathbb{P}$.4.2 clearly holds since $\phi(x)$ is homogeneous.

For the satisfaction of property $\mathbb{P}$.5, observe that if $\{x^\nu\} \subset X$ and $\{x^\nu\} \to x^o$ (note $x^o \in X$), and $u^\nu \in \mathbb{K}(x^\nu)$ for $\nu = 1,2, \ldots$, with $\{u^\nu\} \to u^o$, then $u^\nu = \theta^\nu \phi(x^\nu)$, $\nu = 1,2, \ldots$ and $u^o = \phi(x^o)$

where $\{\theta^{\nu}\} \to \theta^{o} \in [0,1]$, and $u^{o} \in \mathbb{K}(x^{o})$. Thus $x \to \mathbb{K}(x)$ is a closed dynamic correspondence.

Property $\mathbb{P}.6$ holds by the definition of $\mathbb{K}(x)$. Concerning property $\mathbb{E}.S$, note first that $\mathbb{E}(u)$ is a subset of ISOQ $\mathbb{L}(u) := \{x \in X : \phi(x) = u\}$. An equality is used for defining ISOQ $\mathbb{L}(u)$ since $\phi(x)$ is homogeneous and strictly increasing for any scalar replication of a vector $x \in X$. Thus we need only show that ISOQ $\mathbb{L}(u)$ is totally bounded. Now consider $\{x^{\mu}\} \subset X$ with $\phi(x^{\mu}) = u$, $(\mu = 1,2, \ldots)$ for $\mathbb{L}(u)$ not empty. Then

$$x_j^{\mu}(t) = x_{ji}^{\mu} = \lambda_i^{\mu} c_i^{(j)} \ , \ t \in [(i-1), i) \ , \ i \in \{1,2, \ldots, N\} \ ,$$

$$x_j^{\mu}(t) = x_{j(N+1)}^{\mu} = \lambda_{N+1}^{\mu} c_{N+1}^{(j)} \ , \ t \in [N,+\infty) \ , \ c_i^{(j)} \in R_+ \ \forall i, j$$

with

$$u_i = \phi_i^{\mu} = \sum_{j=1}^{n} \beta_{ij} \sum_{\nu=(i+1-\kappa_o)}^{i} \alpha_j(i+1-\nu) x_j^{\mu}(\nu)$$

for $i = \kappa_o, \kappa_o + 1, \ldots, N, (N+1)$, and $\mu \in \{1,2, \ldots\}$. Accordingly for $\nu = 1,2, \ldots$ and $j \in \{1,2, \ldots, n\}$

$$x_j^{\mu}(i) \leqq \frac{u_{\kappa_o}}{\beta_{\kappa_o j} \alpha_j(\kappa_o + 1 - i)} \ , \ i = 1,2, \ldots, \kappa_o$$

$$x_j^{\mu}(i) \leqq \frac{u_i}{\beta_{ij} \alpha_j(1)} \ , \ i = (\kappa_o + 1), \ldots, N, (N+1) \ .$$

Thus for all $(\mu = 1,2, \ldots)$ $x \in \mathrm{ISOQ}\ \mathbb{L}(u)$ only if uniformly in μ

$$0 \leqq x_j^{\mu}(i) \leqq x_j^{o}(i) \ , \ (i = 1,2, \ldots, (N+1)) \text{ and } j \in \{1,2, \ldots, n\} \ .$$

Consequently, since there are a finite number of intervals $[(i-1),i)$, there exists a subsequence of $\left\{x_j^{\mu}\right\}$ for each $j \in \{1,2, \ldots, n\}$ such that $\left\{x_j^{\mu_j}\right\} \to x_j^o$, and $x^o = \left(x_1^o, \ldots, x_n^o\right) \in \mathrm{ISOQ}\ \mathbb{L}(u)$. Thus $\mathrm{ISOQ}\ \mathbb{L}(u)$ is compact. Since $\mathbb{E}(u) \subset \mathrm{ISOQ}\ \mathbb{L}(u)$, $\mathbb{E}(u)$ is relatively comapct or totally bounded. Thus property $\mathbb{E}$.S holds for $x \to \mathbb{K}(x)$, and of necessity $\mathbb{E}$ holds although it is clear directly that this property holds.

To continue with the properties of $x \to \mathbb{K}(x)$ property $\mathbb{P}$.T.1 holds since $\phi_i = 0$ for $(1 \leqq i \leqq (\kappa_o - 1))$. Also, since $\beta_{ij} = 0$ for $i \geqq \bar{t}_{x_j}$, $j \in \{1,2, \ldots, n\}$, it follows from the definition of $\boldsymbol{\phi}(x)$ that $\mathbb{P}$.T.2 holds. From the relation defining ϕ_i, it is clear that if u is summable, x must be summable in each component. Thus property $\mathbb{L}$.T.1 holds. Finally, if $\bar{t}_u = t^o < +\infty$ and $x_j(t) > 0$ for $t \geqq t^o$ with $u(t) = 0$ for $t \geqq t^o$, and $x \in \mathbb{L}(u)$, u may be obtained by setting $x_j(t) = 0$ for $t \geqq t^o$, since such inputs do not affect the positive values of u. Therefore $\mathbb{L}$.T.2 holds.

A dynamic CES-type scalar valued production function is obtained if the definition $\boldsymbol{\phi}(x)$ is altered so that

$$\phi_i := \begin{cases} 0 \text{ for } 1 \leqq i \leqq (\kappa_o - 1) \ , \ \kappa_o > 1 \\ \left[\sum_{j=1}^{n} \beta_{ij}(X_{ij})^{-\rho}\right]^{-\frac{1}{\rho}} , \ i \in \{\kappa_o, \kappa_o + 1, \ldots, N, N+1\} \end{cases}$$

where

$$X_{ij} = \sum_{\nu=(i+1-\kappa_o)}^{i} \alpha_j(i+1-\nu)x_j(\nu) \qquad \begin{array}{l} i \in \{\kappa_o, \kappa_o + 1, \ldots, N, N+1\} \\ j \in \{1,2, \ldots, n\} \end{array}$$

and $(-1 \leqq \rho < 0)$. Here the efficient subsets $\mathbb{E}(u)$ are totally bounded, as distinguished from (2.3-18) which has only bounded efficient subsets $\mathbb{E}(u)$.

The foregoing examples afford a basis for dynamically generalizing certain well known production functions in the steady state case. The correspondence $x \to \mathbb{K}(x)$ defined by (2.3-1), (2.3-2), ..., (2.3-7) affords a ready basis for dynamically generalizing the familiar C.E.S. production function. Let

$$\text{(2.3-16)} \qquad X_{j\nu}^{(i)} := \begin{cases} \sum_{\tau=\left(\nu+1-\nu_i^o\right)}^{\nu} \alpha_j^{(i)}(\nu+1-\tau)x_j(\tau) \ , \ \nu \geqq \nu_i^o \\ 0 \ , \ 1 \leqq \nu \leqq \left(\nu_i^o - 1\right) \ , \ \nu_i^o > 1 \ , \end{cases}$$

for $i \in \{1,2, \ldots, m\}$, $j \in \{1,2, \ldots, n\}$. Then a dynamic C.E.S. correspondence $x \to CES(x)$, $x \in (L_\infty)_+^n$, $CES(x) \in 2^{(L_\infty)_+^m}$, is defined by

$$\text{(2.3-17)} \quad CES(x) := \left\{u \in (L_\infty)_+^m : u_i = \theta w_i \ , \ i \in \{1,2, \ldots, m\} \ , \ \theta \in [0,1]\right\}$$

where $w := (w_1, w_2, \ldots, w_m)$ is given by $w_i(t) = w_{i\nu}$, $t \in [\nu - 1, \nu)$, with

$$(2.3\text{-}18)\qquad w_{i\nu} = \left[\sum_{j=1}^{n} \left\{\beta_{j\nu}^{(i)}\left(X_{j\nu}^{(i)}\right)^{-\rho_i}\right\}\right]^{-\frac{1}{\rho_i}} , \quad \begin{array}{l} \nu \geqq \nu_i^o \\ i \in \{1,2, \ldots, m\} \end{array}$$

in which

$$(2.3\text{-}19)\qquad \rho_i \geqq -1 \ , \ \rho_i \neq 0 \quad i \in \{1,2, \ldots, m\} \ .$$

In case $m = 1$, the dynamic C.E.S. production function is given by $x \in (L_\infty)_+^n \to u \in (L_\infty)_+$ where $u(t) = u_\nu$, $t \in [(\nu - 1),\nu)$, $\nu = 1,2, \ldots$, and

$$(2.3\text{-}20)\qquad u_\nu = \left[\sum_{j=1}^{n} \left\{\beta_{j\nu}(X_{j\nu})^{-\rho}\right\}\right]^{-\frac{1}{\rho}} \quad \nu = 1,2, \ldots \ .$$

In the expression (2.3-20) time histories of inputs $x_j(t)$, $t \in [0,+\infty)$, $j \in \{1,2, \ldots, n\}$ are related by the formulas (2.3-16) to a time history $u(t)$ on $t \in [0,+\infty)$ of a single output, as in the usual case. However, by retaining $i = (1,2, \ldots, m)$ one obtains a vector valued production function

$$(2.3\text{-}21)\qquad u_{i\nu} = \left[\sum_{j=1}^{n} \left\{\beta_{j\nu}^{(i)}\left(X_j^{(i)}(t)\right)^{-\rho_i}\right\}\right]^{-\frac{1}{\rho_i}} , \quad \begin{array}{l} i \in \{1,2, \ldots, m\} \\ \nu = 1,2, \ldots \end{array}$$

in the sense that (2.3-21) defines the maximal value of the i^{th} output component at the time t. By inclusion of the coefficients $\beta_{j\nu}^{(i)}$, $\nu = 1,2, \ldots$ one may allow for "learning curve" effects which need not be the same for each factor.

Strictly speaking we ought to restrict $-1 \leqq \rho_i < 0$, in order to assure that the efficient subset of input histories is bounded, but this is not adhered to by users of the C.E.S. production function.

The correspondence $u \to \mathcal{M}(u)$ defined by (2.3-9), ..., (2.3-15) provides a basis for a dynamic generalization of the so-called Leontief production function. Let

$$(2.3\text{-}22) \quad U_{j\nu}^{(i)} := \sum_{\tau=\nu}^{\left(\nu+\nu_i^o-1\right)} c_i^{(j)}(\tau + 1 - \nu)u_i(\tau) \ , \ \nu = 1,2, \ldots$$

for $i \in \{1,2, \ldots, m\}$, $j \in \{1,2, \ldots, n\}$. Then a dynamic Leontief-like correspondence $u \to L(u)$, $u \in (L_\infty)_+^m$, $L(u) \in 2^{(L_\infty)_+^n}$, is defined by

$$(2.3\text{-}23) \qquad L(u) := \left\{x \in (L_\infty)_+^n : x \geqq y\right\} \ , \ u \in (L_\infty)_+^m \ ,$$

where $y := (y_1, y_2, \ldots, y_n)$ is given by $y_i(t) = y_{i\nu}$, $t \in [(\nu - 1),\nu)$, with

$$(2.3\text{-}24) \quad y_{j\nu} = \sum_{i=1}^{m} b_{i\nu}^{(j)} \cdot U_{j\nu}^{(i)} \ , \ (\nu = 1,2, \ldots) \ , \ j \in \{1, \ldots, n\} \ .$$

In case $m = 1$, the dynamic Leontief-like production correspondence becomes $u \in (L_\infty)_+ \to L(u) \in 2^{(L_\infty)_+^n}$ where

$$(2.3\text{-}25) \quad x_{j\nu} \geqq b_\nu^{(j)} \cdot \sum_{\tau=\nu}^{\left(\nu+\nu_i^o-1\right)} c^{(j)}(\tau + 1 - \nu)u(\tau) \ , \ \nu = 1,2, \ldots$$

for $j = 1,2, \ldots, n$. If constant rate output history is used, with constant rate input histories, and no "learning effect" is included,

the expression (2.3-25) reduces to

$$x_j \geqq B_j u \quad \left(x_j \in R_+^n\, ,\ u \in R_+\, ,\ B_j \in R_{++}\right) \tag{2.3-26}$$

and the corresponding Leontief Production Function is

$$u = \underset{j}{\mathrm{Min}} \left\{ \frac{x_j}{B_j} \right\} . \tag{2.3-27}$$

When the dynamic C.E.S. production function is made static in the same way, it becomes the familiar form

$$u = \left[\sum_{j=1}^{n} A_j x_j^{-\rho} \right]^{-\frac{1}{\rho}} .$$

2.4 Independence of the Axioms

The axioms stated in Sections 2.1.1, 2.1.3, 2.2.2 and 2.2.4 provide certain alternative systems as a foundation for the theory of dynamic production correspondences. In the preceding section they have been shown to be free of contradiction. However this does not guarantee that they are independent in the following sense:

Definition:

An axiom of a system S is independent in S if and only if there exists a dynamic correspondence $\mathbb{K} : (L_\infty)_+^n \to 2^{(L_\infty)_+^m}$ such that the axiom is not fulfilled while all the remaining ones are satisfied.

<u>Proposition 2.4-1</u>:

The axiom system $S_3 := \{\mathbb{P}.2, \mathbb{P}.3, \mathbb{P}.4.1, \mathbb{P}.4.2, \mathbb{P}.5, \mathbb{P}.6S, \mathbb{E}, \mathbb{P}.T.1, \mathbb{P}.T.2, \mathbb{L}.T.1, \mathbb{L}.T.2\}$ contains independent axioms with respect to S_3 .

<u>Proof</u>:

(a) Property $\mathbb{P}.1$ is not included in S_3 , because clearly $\mathbb{P}.T.2$ cannot hold, if $\mathbb{P}.1$ does not hold, and $\mathbb{P}.1$ would not be independent of $\mathbb{P}.T.2$ in S_3 if $\mathbb{P}.1$ were included in S_3 .

(b) Concerning property $\mathbb{P}.2$, consider the dynamic correspondence $x \in (L_\infty)^n_+ \to \mathbb{K}_2(x) \in 2^{(L_\infty)^m_+}$ defined by

$$\mathbb{K}_2(x) = \begin{cases} \left\{u \in (L_\infty)^m_+ : u_i = \theta w_i \ , \ \theta_i \in [0,1] \ , \ i \in \{1,2, \ldots, m\}\right\} \\ \qquad\qquad \text{for } \sum_1^m w_i < \xi \\ (\hat{L}_\infty)^m_+ \quad \text{for } \sum_1^m w_i \geqq \xi \end{cases} \tag{2.4-1}$$

where the functions w_i are defined as in Proposition (2.3-1) by (2.3-2) $\cdots$ (2.3-7) with $\nu_i^o = \nu^o$ for $i \in \{1, \ldots, m\}$ and

$$\xi : \xi(t) = \begin{cases} 0 & \text{for } t \in [0,\nu^o) \\ > 0 & \text{for } t \in [\nu^o,+\infty) \end{cases} \tag{2.4-2}$$

is a summable function on $[0,+\infty)$. The set $(\hat{L}_\infty)^m_+$ is defined by:

$$(\hat{L}_\infty)^m_+ := \{u \in U : u_i(t) = 0 \ , \ t \in [0,\nu^o) \ , \ i \in \{1,2, \ldots, m\}\}$$

(2.4-3) $$U := \Big\{u \in (L_\infty)^m_+ : u_i(t) = u_i(\nu) \ , \ u_i(\nu) \in R_+ \ , t \in I_\nu = [(\nu-1),\nu) \ ,$$
$$\nu = 1,2, \ldots , i \in \{1,2, \ldots, m\}\Big\} \ .$$

Clearly property $\mathbb{P}.2$ does not hold since $(\hat{L}_\infty)^m_+$ is unbounded. In regard to property $\mathbb{P}.3$ we have two situations to consider: (i) when $\sum_1^m w_i < \xi$ and $\lambda \in [1,+\infty)$, $w_i(t \mid \lambda x) = \lambda w_i(t \mid x) \geqq w_i(t \mid x)$ and $\sum_1^m w_i(x) \leqq \sum_1^m w_i(\lambda x)$. Either $\sum_1^m w_i(\lambda x) \geqq \xi$, in which case any $u \in \mathbb{K}_2(x)$ belongs to $\mathbb{K}_2(\lambda x) = (\hat{L}_\infty)^m_+$, or $\sum_1^m w_i(\lambda x) < \xi$. In the latter case, if $x \notin Y$, $\lambda x \notin Y$ and $\mathbb{K}_2(x) = \mathbb{K}_2(\lambda x)$, and we need only consider $x \in Y$, $\lambda x \in Y$. Then, if $u_i = \theta_i w_i(x)$, $\theta_i \in [0,1]$, $i \in \{1,2, \ldots, m\}$, then $u_i = (\theta_i/\lambda) \cdot w_i(\lambda x)$ where $(\theta_i/\lambda) \in [0,1]$ for all i , and $\mathbb{P}.3$ holds, (ii) when $\sum_1^m w_i(x) \geqq \xi$, $\sum_1^m w_i(\lambda x) \geqq \xi$ and $\mathbb{K}_2(x) = \mathbb{K}_2(\lambda x)$. Thus property $\mathbb{P}.3$ holds in all cases. See Proposition (2.3-1) for Y .

Now obviously $\mathbb{P}.4.1$ holds when $x \in Y$ and $x >> 0$. Concerning $\mathbb{P}.4.2$, when $\sum_1^m w_i < \xi$, one need only take $\lambda_\theta \geqq \theta$, and when $\sum_1^m w_i \geqq \xi$ only scalar $\lambda_\theta \geqq 1$ will suffice.

For the closure of the correspondence $x \to \mathbb{K}_2(x)$, i.e., property $\mathbb{P}.5$, let $\{x^\alpha\} \to x^o$, $u^\alpha \in \mathbb{K}_2(x^\alpha)$ for $\alpha = 1,2, \ldots$ and $\{u^\alpha\} \to u^o$. Now either (i) $x^\alpha \in Y$ for almost all α , (ii) $x^\alpha \notin Y$ for almost all α , or (iii) there are two infinite subsequences of $\{\alpha_\kappa\}$, $\{\beta_\kappa\}$ such that $x^{\alpha_\kappa} \in Y$ for all κ and $x^{\beta_\kappa} \notin Y$ for all κ . In the third case the two subsequences $\left\{x^{\alpha_\kappa}\right\}$, $\left\{x^{\beta_\kappa}\right\}$ must converge to the same limit x^o ,

since $\{x^{\alpha}\} \to x^{o}$. But for $\left\{x^{\beta_{\kappa}}\right\}$, $u^{\alpha_{\kappa}} \to 0$ which certainly belongs to $\mathbb{K}_2(x^o)$, while the case for the sequence $\left\{x^{\alpha_{\kappa}}\right\}$ is the same as that of the first case. In the second case, $\{u^{\alpha}\} \to 0 \in (L_{\infty})^m_+$ and $0 \in \mathbb{K}_2(x^o)$. In the first case $x^o \in Y$ and $u^o \in U$, and either $\sum_{i=1}^{m} w_i(x^o) < \xi$ or $\sum_{i=1}^{m} w_i(x^o) \geqq \xi$. If the latter of these two inequalities holds, $\mathbb{K}(x^o) = (\hat{L}_{\infty})^m_+$ and $u^o \in \mathbb{K}(x^o)$ because, as the limit of a sequence of step functions u^{α} , with $u_i^{\alpha}(t) = 0$ for $t \in [0,\nu^o)$, u^o has these same properties and hence belongs to $(\hat{L}_{\infty})^m_+$. When the first inequality holds $u_i^{\alpha} = \theta_i w_i(x^{\alpha})$, $i \in \{1, \ldots, m\}$, $\alpha = 1,2, \ldots$, and there exists a subsequence $\{\alpha_{\kappa}\} \subset \{\alpha\}$ such that $\left\{\theta_i^{\alpha_{\kappa}}\right\} \to \theta_i^o$, $\theta_i^o \in [0,1]$, $i \in \{1,2, \ldots, m\}$, and $u_i^o = \theta_i^o w_i(x^o)$, implying $u^o \in \mathbb{K}(x^o)$. Thus property $\mathbb{P}$.5 holds.

Concerning property $\mathbb{P}$.6, when the vector $x \notin Y$, $\mathbb{K}(x) = 0 \in (L_{\infty})^m_+$ and there is no issue. If $x \in Y$ and $\sum_{1}^{m} w_i(x) < \xi$, property $\mathbb{P}$.6 follows merely from the definition (2.3-1). If $x \in Y$ and $\sum_{1}^{m} w_i(x) \geqq \xi$, property $\mathbb{P}$.6 still holds, because if $u \in (\hat{L}_{\infty})^m_+$, $(\theta_1 u_1, \ldots, \theta_m u_m)$ likewise is a step function with $u_i(t) = 0$, $t \in [0,\nu^o)$, $i \in \{1,2, \ldots, m\}$.

Consider now the property $\mathbb{E}$, for which there are two cases to consider: (i) $\sum_{1}^{m} u_i < \xi$, (ii) $\sum_{1}^{m} u_i \geqq \xi$, with $u \in U$. If $u \notin U$, $\mathbb{L}(u)$ is empty with $\mathbb{E}(u)$ empty and bounded, because there are no efficient vectors $x \in \mathbb{L}(u)$. It is necessary for $x \in \mathbb{E}(u)$, $u \in U$, that $x \in \mathbb{L}(u)$ and x is at minimal ray distance from the null vector. In the first case, i.e., $\sum_{1}^{m} u_i < \xi$, $\mathbb{L}(u)$ is not empty, and $x \in \mathbb{E}(u)$ must satisfy $w_i(x) = u_i$, $i \in \{1,2, \ldots, m\}$. From relations

(2.3-2) $\cdots$ (2.3-7) defining the functions $w_i(x)$, it follows that $||x|| < +\infty$. In the second case, when $x \in \mathbb{E}(u)$ it is necessary that $\sum_1^m w_i(x) = \xi$ and since $||\xi|| < +\infty$ it follows from (2.3-2) $\cdots$ (2.3-7) that $||x|| < +\infty$. Hence $\mathbb{E}$ is satisfied.

Property $\mathbb{P}$.T.1 holds for the correspondence $x \to \mathbb{K}_2(x)$ by the definition (2.-2) of the functions w_i in case $\sum_1^m w_i < \xi$ for $x \in Y$, and by the definition of the set $\mathbb{K}_2(x)$ in case $\sum_1^m w_i \geqq \xi$ and $x \in Y$. If $x \notin Y$, $u_i = 0$, $i \in \{1,2, \ldots, m\}$ and $\mathbb{P}$.T.1 is trivially satisfied.

Property $\mathbb{P}$.T.2 holds due to the property (2.3-6) of the functions $\beta_j^{(i)}$, $j \in \{1,2, \ldots, n\}$, $i \in \{1,2, \ldots, m\}$. In the case of property $\mathbb{L}$.T.1, if u is summable in each component and $\sum_1^m u_i < \xi$, it follows from (2.3-2) $\cdots$ (2.3-7) that the equations $u_i = w_i$, $i \in \{1, \ldots, m\}$ may be satisfied by a vector x summable in each component. Thus $\mathbb{L}$.T.1 holds when $\sum_1^m u_i < \xi$. If $\sum_1^m u_i \geqq \xi$, a vector x satisfying $\sum_1^m w_i(x) = \xi$ may yield u , and since ξ is summable it follows again from (2.3-2) $\cdots$ (2.3-7) that x may be summable in each component to obtain u . Finally, it is clear from the property (2.3-6) for the functions $\beta_j^{(i)}$ that property $\mathbb{L}$.T.2 holds.

Thus the axiom $\mathbb{P}$.2 is independent of the other axioms of the set S_3 .

(c) Next, concerning property $\mathbb{P}$.3, consider the dynamic correspondence $x \in (L_\infty)_+^n \to \mathbb{K}_3(x) \in 2^{(L_\infty)_+^m}$

$$
(2.4\text{-}4) \qquad
\begin{aligned}
w_i &:= \begin{cases} 0 \in (L_\infty)_+^m & \text{if } \|x\| = 0 \text{ or } x \notin Y \\ w_{i\nu} \in R_+ & \text{for } x \in Y,\ t \in I_\nu,\ \nu = 1,2,\ \ldots \end{cases} \\
w_{i\nu} &:= \begin{cases} 0 \quad \text{for } 1 \leqq \nu \leqq \left(\nu_i^o - 1\right),\ \nu_i^o > 1 \\ \displaystyle\sum_{j=1}^{n} \beta_{j\nu}^{i} \Delta(\|x\|) \sum_{\tau=\left(\nu+\nu_i^o-1\right)}^{\nu} \alpha_j^{(i)}(\nu+1-\tau) \frac{x_j(\tau)}{\|x\|},\ \nu \geqq \nu_i^o \end{cases}
\end{aligned}
$$

where the coefficients $\alpha_j^{(i)}(\sigma)$ and the coefficient functions $\beta_j^{(i)}$ satisfy (2.3-3) $\cdots$ (2.3-5) and (2.3-6), (2.3-7) respectively, and $\Delta(\sigma)$ is a continuous, nonnegative function with $\lim\limits_{\sigma\to 0} \frac{\Delta(\sigma)}{\sigma} = 1$ and graph as shown in Figure 1.

Then clearly property $\mathbb{P}.3$ does not hold, because there exists an $x \in Y$ with $\sigma_0 \leqq \|x\| \leqq \sigma_1$ and a scalar $\lambda \in [1,+\infty)$ such that $\Delta(\|\lambda x\|) < \Delta(\|x\|)$, implying that $w_i(t \mid \lambda x) < w_i(t \mid x)$, $x \in Y$, $i \in \{1,2,\ \ldots,\ m\}$. Hence $\mathbb{K}_3(\lambda x) \not\supset \mathbb{K}_3(x)$ for all $x \in (L_\infty)_+^n$ and $\lambda \in [1,+\infty)$.

However, property $\mathbb{P}.2$ holds since $\|w_i\| < +\infty$ for $\|x\| < +\infty$. Property $\mathbb{P}.4.1$ holds for $x >> 0$. In regard to property $\mathbb{P}.4.2$, consider $\|x\| > 0$. From the graph of $\Delta(\sigma)$ it is clear that there exists for each $\theta \in (0,+\infty)$ a scalar λ_θ such that $\Delta(\|\lambda_\theta x\|) \geqq \theta\Delta(\|x\|)$ and $w_i(t \mid \lambda_\theta x) \geqq \theta w_i(t \mid x)$, $t \in [0,+\infty)$, whence if $u \in \mathbb{K}_3(x)$, $(\theta u) \in \mathbb{K}_3(\lambda_\theta x)$ for $\theta \in (0,+\infty)$. Thus property $\mathbb{P}.4.2$ holds.

Concerning the closure of the dynamic correspondence $x \to \mathbb{K}_3(x)$, the argument given for property $\mathbb{P}.5$ in Proposition (2.3-1) applies here and $\mathbb{P}.5$ holds, and property $\mathbb{P}.6S$ follows from the definition (2.3-1).

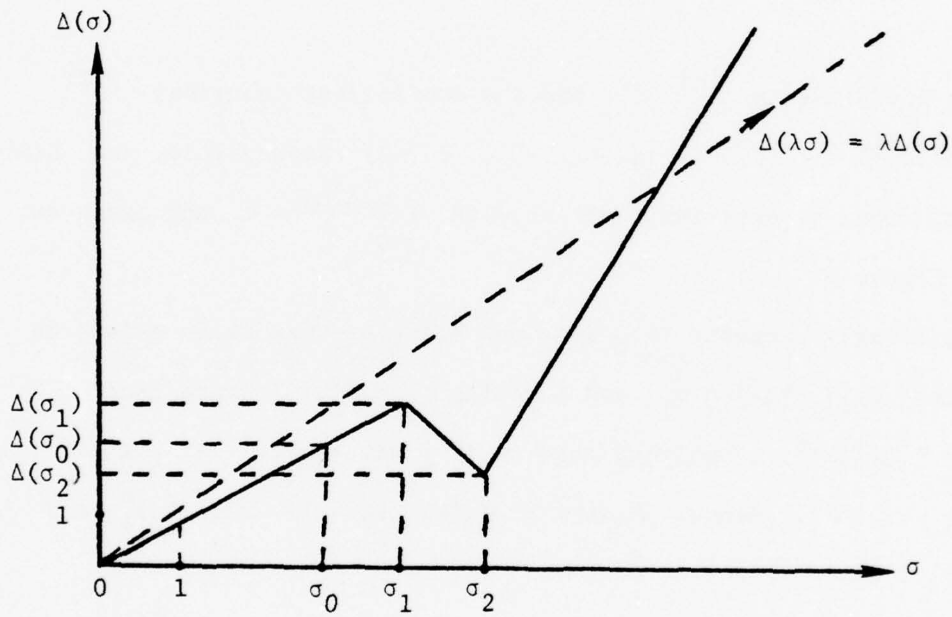

FIGURE 1

GRAPH OF THE FUNCTION $\Delta(\sigma)$,

$\sigma_0 > \operatorname*{Max}_i \left\{\nu_i^o\right\}$, $\Delta(\sigma) \to +\infty$ AS $\sigma \to +\infty$

We note also that $\mathbb{P}.6$ holds merely by taking $\theta_i = \theta$, $i \in \{1,2, \ldots, m\}$.

The argument used for property $\mathbb{E}$ in Proposition (2.3-1) applies here and $\mathbb{E}$ holds for $x \to \mathbb{K}_3(x)$. Similarly, the arguments for properties $\mathbb{P}.T.1$, $\mathbb{P}.T.2$, $\mathbb{L}.T.1$ and $\mathbb{L}.T.2$ apply, and these properties are likewise fulfilled by $x \to \mathbb{K}_3(x)$.

Thus the axiom $\mathbb{P}.3$ is independent of the other axioms of the system.

(d) Next consider the axiom $\mathbb{P}.4.1$, this property is negated by changing the definition of the functions w_i $(i = 1,2, \ldots, m)$ in the correspondence $x \to \mathbb{K}(x)$ of Proposition (2.3-1) so that equations (2.3-2) hold for $i \in \{1,2, \ldots, (m - 1)\}$ with $w_m(t) = 0$, $t \in [0,+\infty)$. Then clearlv nothing is altered in the arguments for all properties of S_3 except $\mathbb{P}.4.1$ which is negated. Hence axiom $\mathbb{P}.4.1$ is independent of the other axioms of S_3 .

(e) In the case of axiom $\mathbb{P}.4.2$, consider the dynamic correspondence $x \in (L_\infty)^m_+ \to \mathbb{K}_4(x) \in 2^{(L_\infty)^m_+}$, where

$$\mathbb{K}_4(x) = \Big\{u \in (L_\infty)^m_+ : u_i = \theta_i w_i \ , \ \theta_i \in [0,1] \ , \ i \in \{1,2, \ldots, m\} \ , \quad ||w_i|| \leqq A \in R_{++}\Big\} \tag{2.4-6}$$

with the functions w_i defined by (2.3-2) ⋯ (2.3-7) as in Proposition (2.3-1). Then clearly property $\mathbb{P}.4.2$ cannot hold. However property $\mathbb{P}.2$ holds outright since $||\theta_i w_i|| = A$ for any $x \in (L_\infty)^n_+$, and property $\mathbb{P}.3$ holds since $w_i(t \mid \lambda x) \geqq w_i(t \mid x)$ for $\lambda \in [1,+\infty)$ and $t \in [0,+\infty)$. Property $\mathbb{P}.4.1$ is satisfied as before when $x >> 0$.

Regarding property $\mathbb{P}.5$, if $\{x^\alpha\} \to x^o$, $u^\alpha \in \mathbb{K}_4(x^\alpha)$ for $\alpha = 1,2, \ldots$ and $\{u^\alpha\} \to u^o$, it follows that $u_i^\alpha = \theta_i^\alpha w_i(x^\alpha)$, $||\theta_i^\alpha w_i(x^\alpha)|| = \theta_i^\alpha ||w_i(x^\alpha)|| \leqq A$, $\theta_i \in [0,1]$ for $\alpha = 1,2, \ldots$, and $i \in \{1,2, \ldots, m\}$. Thus there exists a subsequence $\{\alpha_k\} \subset \{\alpha\}$ such that $\left\{\theta_i^{\alpha_k}\right\} \to \theta_i^o$, $i \in \{1,2, \ldots, m\}$, with $\theta_i^o ||w_i(x^o)|| \leqq A$. Simultaneously, $\left\{u_i^\alpha\right\} \to u_i^o$ and $\left\{w_i(x^\alpha)\right\} \to w_i(x^o)$ for $i \in \{1,2, \ldots, m\}$. Consequently $u_i^o = \theta_i^o w_i^o$ where $\theta_i^o \in [0,1]$ and $||\theta_i^o w_i(x^o)|| \leqq A$ and $u^o \in \mathbb{K}(x^o)$. Hence property $\mathbb{P}.5$ holds.

Property $\mathbb{P}.6S$ is guaranteed for the correspondence $x \to \mathbb{K}_4(x)$ by the definition (2.4-6).

The arguments used in Proposition (2.3-1) to show that properties $\mathbb{E}$, $\mathbb{P}.T.1$, $\mathbb{P}.T.2$, $\mathbb{L}.T.1$, $\mathbb{L}.T.2$ hold, carry over for the correspondence $x \to \mathbb{K}_4(x)$.

Hence axiom $\mathbb{P}.4.2$ is independent of the other axioms of S.3.

(f) Next we consider the negation of property $\mathbb{P}.5$ by considering the dynamic correspondence $x \in (L_\infty)_+^n \to \mathbb{K}_5(x) \in 2^{(L_\infty)_+^m}$ defined by:

$$\text{(2.4-7)} \quad \mathbb{K}_5(x) := \begin{cases} \left\{u \in (L_\infty)_+^m : u_i = \theta_i w_i \, , \, \theta_i \in [0,1) \, , \, i \in \{1,2, \ldots, m\}\right\} \\ \qquad\qquad\qquad \text{for } x \neq 0 \\ \{0\} \text{ for } x = 0 \end{cases}$$

with the functions w_i defined as in Proposition (2.3-1) by (2.3-2) $\cdots$ (2.3-7). The arguments of Proposition (2.3-1) may be used to show that all properties of S_3 hold except $\mathbb{P}.5$. In order to see that property $\mathbb{P}.5$ need not be satisfied,

consider the infinite sequence

$$\{x^\alpha\} \to x^o \qquad \alpha = 1,2, \ldots$$

$$u^\alpha = \left(\frac{\alpha}{\alpha + 1} w_1^o, 0, 0, \ldots, 0\right), \alpha = 1,2, \ldots$$

where $w_1^o : w_1^o(t) = w_1(t \mid x^o)$, $t \in [0,+\infty)$, and $\{x^\alpha\} \to x^o$, $u^\alpha \in \mathbb{K}_5(x^\alpha)$ for $\alpha = 1,2, \ldots$ with $\{u^\alpha\} \to \left(w_1^o, 0, 0, \ldots, 0\right)$. Since $\left(w_1^o, 0, \ldots, 0\right) \notin \mathbb{K}_5(x^o)$, $\mathbb{P}.5$ does not hold.

(g) For the failure of property $\mathbb{P}.6$ consider the dynamic correspondence $x \in (L_\infty)_+^n \to \mathbb{K}_6(x) \in 2^{(L_\infty)_+^m}$ defined by

$$\text{(2.4-8)} \qquad \mathbb{K}_6(x) := \begin{cases} \{0\} \in (L_\infty)_+^m & \text{for } \sum_1^m w_i < \xi_1 \\ \left\{u \in (L_\infty)_+^m : \sum_1^m u_i = \xi_2\right\} \cup \{0\} & \text{for } \xi_1 \leqq w_i < \xi_2 \\ \left\{u \in (L_\infty)_+^m : \sum_1^m u_i \leqq \sum_1^m w_i + \xi_2\right\} & \text{for } \sum_1^m w_i \geqq \xi_2 \end{cases}$$

where the functions w_i are defined as in Proposition (2.3-1) by (2.3-2) $\cdots$ (2.3-7), with $\nu_i^o = \nu^o$ for $i \in \{1,2, \ldots, m\}$ and ξ_2 given by

$$\text{(2.4-9)} \qquad \xi_i(t) := \begin{cases} 0 & \text{for } t \in [0,\nu^o),\ \nu^o > 0 \\ c_i \in R_{++} & \text{for } t \in \left[\nu^o, \bar{t}_x\right] \\ 0 & \text{for } t \in (\bar{t}_x, +\infty) \end{cases}$$

for $i \in \{1,2\}$ and $c_2 > c_1$. Then property $\mathbb{P}.6$ does not hold, because, for $x \in Y$ where $\xi_1 \leqq \sum_1^m w_i(x) < \xi_2$, there exists a vector $u \in \mathbb{K}_6(x)$ with $\sum_1^m u_i = \xi_2$ and for some $\theta_i \in (0,1)$ we may have $0 < \sum_1^m (\theta_i u_i) < \xi_2$, but $(\theta_1 u_1, \ldots, \theta_m u_m)$ does not belong to $\mathbb{K}_6(x)$. Thus property $\mathbb{P}.6$ does not hold.

Since $||w_i|| < +\infty$ for $||x|| < +\infty$ and the function ξ_2 is bounded, property $\mathbb{P}.2$ is satisfied. Concerning property $\mathbb{P}.3$, if x yields $\sum_1^m w_i(x) < \xi_1$, then $\mathbb{K}_6(x) = \{0\}$ which is a subset of all output sets $\mathbb{K}_6(x)$ for $x \in (L_\infty)_+^m$ including those where the vector of input functions is of the form (λx) , $\lambda \in [1,+\infty)$. If $\xi_1 \leqq \sum_1^m w_i(x) < \xi_2$, either $u = 0$ or $\sum_1^m u_i = \xi_2$. The case $u = 0$ need not concern us. Hence suppose $\sum_1^m u_i = \xi_2$. For $\lambda \in [1,+\infty)$, either $\xi_1 \leqq \sum_1^m w_i(\lambda x) < \xi_2$ and $\mathbb{K}_6(\lambda x) = \mathbb{K}_6(x)$, or $\sum_1^m w_i(\lambda x) \geqq \xi_2$. In the latter case, $\sum_1^m u_i(x) = \xi_2 \leqq \sum_1^m w_i(\lambda x) + \xi_2$ and $u \in \mathbb{K}_6(\lambda x)$. Thus, in all cases $\mathbb{K}_6(x) \subset \mathbb{K}_6(\lambda x)$ for $\lambda \in [1,+\infty)$ and property $\mathbb{P}.3$ holds.

Turning to property $\mathbb{P}.4.1$, it is clear from the properties of the functions w_i that there exists $x \in Y$, $x >> 0$ such that $\xi_1 \leqq \sum_1^m w_i(x) < \xi_2$ and the output vector $u = \left(\frac{\xi_2}{m}, \frac{\xi_2}{m}, \ldots, \frac{\xi_2}{m}\right)$ is possible.

For property $\mathbb{P}.4.2$ we need concern ourselves only with vectors $x \in Y$, such that $\sum_1^m w_i(x) \geqq \xi_1$. Consider first the case where

$\xi_1 \leqq \sum_1^m w_i(x) < \xi_2$. Then, if $u \in \mathbb{K}_6(x)$ with $u \neq 0$, we have $\sum_1^m u_i = \xi_2$. In case $\theta \in (0,1]$, $(\theta u) \in \mathbb{K}_6(\lambda_\theta x)$ when

$$\lambda_\theta \geqq \left(c_2 / \operatorname*{Inf}_t \left\{\sum_1^m w_i(t) : t \in \left[\nu^o, \bar{t}_x\right]\right\}\right) ,$$

and when $\theta \in (1,+\infty)$, $(\theta u) \in \mathbb{K}_6(\lambda_\theta x)$ when

$$\lambda_\theta \geqq \left((\theta - 1)c_2 / \operatorname*{Inf}_t \left\{\sum_1^m w_i(t) : t \in \left[\nu^o, \bar{t}_x\right]\right\}\right) .$$

In the second case when $\sum_1^n w_i(x) \geqq \xi_2$, $u \in \mathbb{K}_6(x)$ if $\sum_1^m u_i \leqq \sum_1^m w_i(x) + \xi_2$. Then for $\theta \in (0,1]$, $\lambda_\theta = 1$ suffices for $u \in \mathbb{K}(\lambda_\theta x)$. For $\theta \in (1,+\infty)$, two cases arise: either $\sum_1^m (\theta u_i) \leqq \xi_2$, in which case $\lambda_\theta = 1$ suffices, or $\sum_1^m (\theta u_i) > \xi_2$ and

$$\lambda_\theta \geqq \left[\left(\theta \sum_1^m u_i - c_2\right) / \operatorname*{Inf}_t \left\{\sum_1^m w_i(t) : t \in \left[\nu^o, \bar{t}_x\right]\right\}\right]$$

yields $(\theta u) \in \mathbb{K}_6(\lambda_\theta x)$. Thus property $\mathbb{P}.4.2$ holds.

For investigation of property $\mathbb{P}.5$ (closure of the correspondence $x \to \mathbb{K}_6(x)$), let $\{x^\alpha\} \to x^o$, $u^\alpha \in \mathbb{K}_6(x^\alpha)$ for $\alpha = 1,2, \ldots$ and $\{u^\alpha\} \to u^o$. There are three cases to consider.

<u>Case 1</u>:

x^o yields $\sum_1^m w_i(x^o) < \xi_1$. Then $\mathbb{K}_6(x^o) = \{0\}$. From the properties (2.3-2) $\cdots$ (2.3-7) for the functions w_i it is clear that there exists

a positive integer N_1 such that $\sum_1^m w_i(x^\alpha) < \xi_1$ for $\alpha \geqq N_1$, implying $u^o \varepsilon \mathbb{K}_6(x^o)$.

Case 2:

x^o yields $\xi_1 \leqq \sum_1^m w_i(x^o) < \xi_2$. Here there exists a positive integer N_2 such that $\sum_1^m w_i(x^\alpha) < \xi_2$ for $\alpha \geqq N_2$ and for each $\alpha \geqq N_2$ either $\sum_1^m u_i^\alpha = \xi_2$ or $\sum_1^m u_i^\alpha = 0$. Since $\{u^\alpha\} \to u^o$, there exists an integer $N_2' \geqq N_2$ such that either $\sum_1^m u_i^o = \xi_2$ or $u^o = 0$. In either case $u^o \varepsilon \mathbb{K}_6(x^o)$.

Case 3:

x^o yields $\sum_1^m w_i(x^o) \geqq \xi_2$. Here there exists a positive integer N_3 such that for $\alpha \geqq N_3$, either $\sum_1^m w_i(x^\alpha) \geqq \xi_2$ or $\xi_1 \leqq \sum_1^m w_i(x^\alpha) < \xi_2$ and for each $\alpha \geqq N_3$, either $\sum_1^m u^\alpha = \xi_2$ or $\sum_1^m u^\alpha = \sum_1^m w_i(x^\alpha) + \xi_2$. Since $\{u^\alpha\} \to u^o$, there exists an integer $N_3' \geqq N_3$ such that either $\sum_1^m u_i^o = \xi_2$ or $\sum_1^m u_i^o = \sum_1^m w_i(x^o) + \xi_2$.

Thus, property $\mathbb{P}.5$ holds for the correspondence $x \to \mathbb{K}_6(x)$.

Concerning the satisfaction of property $\mathbb{E}$, consider $u \varepsilon (L_\infty)_+^m$. If $u = 0$, $\mathbb{E}(u) = \{0\}$ is bounded, (see definition (2.2.3-1)). If $u \geq 0$ and $\mathbb{L}(u)$ is not empty, either $\sum_1^m u_i = \xi_2$ or $\sum_1^m u_i \neq \xi_2$. In the first case,

$$\mathbb{L}(u) = \left\{x \varepsilon (L_\infty)_+^n : \sum_1^m w_i(x) \geqq \xi_1\right\}$$

and in the second case

$$\mathbb{L}(u) = \left\{ x \, \varepsilon \, (L_\infty)^n_+ : \sum_1^m w_i(x) \geqq \xi_2 \right\} .$$

In either case, the equality sign must hold for $x \, \varepsilon \, \mathbb{E}(u)$, since $\mathbb{L}(u)$ is closed (property $\mathbb{P}.5$). It follows from (2.3-2) ··· (2.3-7) defining the functions w_i that $\mathbb{E}(u)$ is bounded, since ξ_1 and ξ_2 are bounded.

Property $\mathbb{P}$.T.1 and $\mathbb{P}$.T.2 hold due to conditions (2.4-9) and (2.3-6), and clearly for the two cases of $\mathbb{L}(u)$, $u \geq 0$ and $\mathbb{L}(u) \neq \emptyset$ above, it follows from the conditions defining the functions w_i that $\mathbb{L}$.T.1 and $\mathbb{L}$.T.2 hold.

Hence it is shown that the axiom $\mathbb{P}$.6S is independent of the others in S.3.

(h) For the failure of the axiom $\mathbb{E}$, consider the dynamic correspondence $x \, \varepsilon \, (L_\infty)^n_+ \rightarrow \mathbb{K}_7(x) \, \varepsilon \, 2^{(L_\infty)^m_+}$ defined by (2.3-1) with relations (2.3-2) modified by

$$(2.4\text{-}10) \quad w_{i\nu} := \begin{cases} 0 \quad \text{for} \quad 1 \leqq \nu \leqq \left(\nu_i^o - 1\right) , \ \nu_i^o > 1 \\ \prod_{j=1}^{n} \beta_{j\nu}^{(i)} \sum_{\tau=\nu+\nu_i^o-1}^{\nu} \alpha_j^{(i)}(\nu + 1 - \tau) x_j(\tau) , \ \nu \geqq \nu_i^o . \end{cases}$$

The technical coefficients $\alpha_j^{(i)}(\sigma)$ and coefficient functions $\beta_j^{(i)}$ are again taken to satisfy (2.3-3) ··· (2.3-7), with $\beta_{j\nu}^{(i)}$ strictly positive for all i, j and ν.

By arguments which exactly parallel those given in Proposition (2.3-1) one may show that all properties of S_3 hold except property $\mathbb{E}$. In the case of property $\mathbb{E}$, consider an input set $\mathbb{L}(u)$ corresponding to a vector $u \geq 0$ belonging to $(L_\infty)_+^m$ such that $\mathbb{L}(u) \neq \emptyset$. It is given by

$$\mathbb{L}(u) = \left\{x \in (L_\infty)_+^m : u \in \mathbb{K}_7(x)\right\}$$

$$= \left\{x \in (L_\infty)_+^m : w_i(x) = \lambda_i u_i \ , \ \lambda_i \in [1,+\infty) \ , \ i \in \{1, \ldots, m\}\right\} .$$

The isoquant of $\mathbb{L}(u)$, i.e., the subset of vectors $x \in \mathbb{L}(u)$ such that $w_i(x) = u_i$. Let x_p and x_q be two distinct input histories of a vector $x \in \text{ISOQ } \mathbb{L}(u)$ with norm $||x||$. Then no matter how large the norm $||x_p||$ is one may choose $||x_q||$ small enough so that $x \in \text{ISOQ } \mathbb{L}(u)$. Thus ISOQ $\mathbb{L}(u)$ is unbounded. Now clearly $\mathbb{E}(u) \subset$ ISOQ $\mathbb{L}(u)$. Further ISOQ $\mathbb{L}(u) \subset \mathbb{E}(u)$, because suppose $x \in \text{ISOQ } \mathbb{L}(u)$ with $y \leq x$. Then for some $j \in \{1,2, \ldots, n\}$, $y_j(\nu) < x_j(\nu)$, $\min_i \nu_i^o \leqq \nu \leqq \bar{t}_{x_j}$. Then for some $i \in \{1,2, \ldots, m\}$, $w_i(y) < w_i(x) = u_i$, implying $y \notin \text{ISOQ } \mathbb{L}(u)$. Hence $\mathbb{E}(u) = \text{ISOQ } \mathbb{L}(u)$ and $\mathbb{E}(u)$ is unbounded. Thus property $\mathbb{E}$ fails.

(i) For the failure of axiom $\mathbb{P}$.T.1 consider the correspondence $x \rightarrow \mathbb{K}(x)$ of Proposition (2.3-1) with the expression for $w_{i\nu}$ in (2.3-2) altered to

$$(2.4\text{-}11) \quad w_{i\nu} := \sum_{j=1}^{n} \beta_{j\nu}^{(i)} \sum_{\tau=(\nu+1-\nu_i^o)}^{\nu} \alpha_j^{(i)}(\nu+1-\tau)x_j(\tau) \ , \ \nu_i^o \geqq 1$$

while otherwise all prescriptions of $x \to \mathbb{K}(x)$ are unaltered. Then by the arguments of Proposition (2.3-1) all properties of S_3 hold, except $\mathbb{P}$.T.1 which fails. Axiom $\mathbb{P}$.T.2 fails for the correspondence $x \to \mathbb{K}(x)$ by deleting condition (2.3-6), leaving all other statements unaltered, while all other properties of S_3 hold. Thus $\mathbb{P}$.T.1 and $\mathbb{P}$.T.2 are each independent of the other properties of S_3 .

(j) For the failure of $\mathbb{L}$.T.1 modify the correspondence of Proposition (2.3-1) so that

$$w_{i\nu} := e^{-\nu} \sum_{j=1}^{n} \beta_{j\nu}^{(i)} \sum_{\tau=\left(\nu+1-\nu_i^o\right)}^{\nu} \alpha_j^{(i)}(\nu+1-\tau)\cdot x_j(\tau) \ , \ \nu \geqq \nu_i^o \tag{2.4-12}$$

while all other statements of (2.3-1) $\cdots$ (2.3-7) hold. Then clearly $\mathbb{L}$.T.1 fails to be satisfied, because even though w_i is summable $e^{\nu}w_{i\nu}$, $\nu = 1,2,\ ...$ is not summable, and it is then implied that x_j is not summable for some $j \in \{1,2,\ ...,n\}$. Otherwise the arguments of Proposition (2.3-1) apply to show that the other properties of S_3 hold. Thus $\mathbb{L}$.T.1 is independent of the other properties of S_3 .

(k) Finally, for the failure of property $\mathbb{L}$.T.2, the correspondence $x \to \mathbb{K}(x)$ of Proposition (2.3-1) may be modified (albeit unrealistically) so that

$$w_{i\nu} := \sum_{j=1}^{n} \beta_{j(\nu+1)}^{(i)} \sum_{\tau=\nu+1-\nu_i^o}^{\nu+1} \alpha_j^{(i)}(\nu + 2 - \tau)x_j(\tau) \ , \ \nu \geqq \nu_i^o \tag{2.4-13}$$

with all other specifications of (2.3-1) ··· (2.3-7) unaltered, and $\alpha_j^{(i)}(1) > 0$ for at least one $i \varepsilon \{1,2, \ldots, m\}$ and $j \varepsilon \{1,2, \ldots, n\}$. Then consider $x \varepsilon \mathbb{L}(u)$, $u \geq 0$, satisfying (2.4-13), i.e., $\theta_i = 1$, where $\bar{t}_u < +\infty$, $\bar{t}_u \varepsilon [\nu - 1,\nu)$ and $x_j(\tau) > 0$ for $\tau \varepsilon [\nu,\nu + 1)$. Then by setting $x_j(\tau) = 0$ for $\tau \varepsilon [\nu,\nu + 1)$, the sum on the right side is diminished and $x \notin \mathbb{L}(u)$ contradicting property $\mathbb{L}$.T.2. The other properties of S_3 still hold, because they do not depend upon the modification made by (2.4-13). Thus property $\mathbb{L}$.T.2 is independent of the others in S_3 .

Sub-Proposition (2.4-1):

The axioms of S_3 are independent for $\mathbb{P}$.3S replacing $\mathbb{P}$.3, $\mathbb{P}$.6 replacing $\mathbb{P}$.6S and $\mathbb{E}$.S (or $\tilde{\mathbb{E}}$.S, or $\tilde{\mathbb{E}}$) replacing $\mathbb{E}$.

The correspondence defined by (2.3-1) ··· (2.3-7) with the alteration (2.4-10) provides a basis for defining a dynamic Cobb-Douglas-like dynamic production correspondence. Let $X_{j\nu}^{(i)}$ be defined as given by (2.3-16). Then the dynamic Cobb-Douglas correspondence $x \varepsilon (L_\infty)_+^n \to CD(x) \varepsilon 2^{(L_\infty)_+^m}$ is expressed by

$$\text{(2.4-14)} \quad CD(x) := \left\{u \varepsilon (L_\infty)_+^m : u_i = \theta_i w_i \ , \ w_i \varepsilon (L_\infty)_+ \ , \ i \varepsilon \{1,2, \ldots, m\}\right\}$$

where

$$\text{(2.4-15)} \quad w_i := \begin{cases} 0 \varepsilon (L_\infty)_+ & \text{if } x \notin Y \\ w_{i\nu} \varepsilon R_+ & \text{for } x \varepsilon Y , t \varepsilon I_\nu = [\nu - 1,\nu) \ , \ \nu = 1,2, \ldots , \end{cases}$$

and

$$w_{i\nu} := \prod_{j=1}^{n} \left(Y_{j\nu}^{(i)}\right)^{a_{ji}} \qquad i \in \{1,2, \ldots, m\} , \nu = 1,2, \ldots \tag{2.4-16}$$

where

$$Y_{j\nu}^{(i)} = \beta_{j\nu}^{(i)} X_{j\nu}^{(i)} ,$$

$$a_{ji} > 0 \quad \text{for} \quad j \in \{1, \ldots, n\} , i \in \{1, \ldots, m\} \tag{2.4-17}$$

$$\sum_{j=1}^{n} a_{ji} = 1 \quad \text{for} \quad i \in \{1,2, \ldots, m\} .$$

A dynamic neoclassical production function is given for $m = 1$, by

$$w_\nu = \prod_{j=1}^{n} (Y_{j\nu})^{a_j} , \nu = 1,2, \ldots , \sum_{j=1}^{n} a_j = 1 . \tag{2.4-18}$$

2.5 Existence of a Dynamic Neoclassical Production Function - Time Substitution

For the purposes of this discussion, consider vectors $x \in (L_\infty)_+^n$ of functions of input rates used for production of a single output, the output rate history of which is denoted by $u \in (L_\infty)_+$. As a scalar function of time output rate is expressed by

$$u : u(t) \in R_+ , t \in [0,+\infty) .$$

The neoclassical (steady state) production function F is a relationship

$$F : x \in R_+^n \to F(x) \in R_+ \tag{2.5-1}$$

between vectors x of constant input rates and the maximal constant output rate obtainable with x. For dynamic studies this relationship is frequently modified by treating the components of x as dependent upom time, i.e., $x(t)$, $t \varepsilon [0,+\infty)$, to which is related a time dependent output rate $u(t)$ expressed by

$$(2.5\text{-}2) \qquad u(t) = F(x(t),t) \quad (\text{or } F(x(t))) \ , \ t \ \varepsilon \ [0,+\infty)$$

which is interpreted as the maximal output rate.

In this fashion a dynamic correspondence

$$(2.5\text{-}3) \quad x \ \varepsilon \ (L_\infty)^n_+ \to \mathbb{P}(x) = \left\{ u \ \varepsilon \ (L_\infty)_+ : u = \theta \, \mathbb{F}(x) \ , \ \theta \ \varepsilon \ [0,1] \right\}$$

is postulated, where in particular the form of $\mathbb{F}(x) \ \varepsilon \ (L_\infty)_+$ is taken as

$$(2.5\text{-}4) \quad \mathbb{F}(x) := \left\{ u \ \varepsilon \ (L_\infty)_+ : u(t) = F(x(t),t) \ , \ t \ \varepsilon \ [0,+\infty) \right\} .$$

Aside from the specialization taken by (2.5-4) where output rate at any time t is related only to input rates at the time t and possibly t as well, which is not essential (see examples of Section 2.3), the function $\mathbb{F}(x)$ of (2.5-3), i.e.,

$$(2.5\text{-}5) \qquad \mathbb{F} : x \ \varepsilon \ (L_\infty)^n_+ \to \mathbb{F}(x) \ \varepsilon \ (L_\infty)_+$$

is here taken as the Dynamic Neoclassical Production Function, with $\mathbb{F}(x)$ having the property that all output histories obtainable from $x \ \varepsilon \ (L_\infty)^n_+$ are of the form $\theta \, \mathbb{F}(x)$, $\theta \ \varepsilon \ [0,1]$, i.e., $\mathbb{F}(x)$ is maximal at each time t for all output histories. For a treatment of the Dynamic Neoclassical Production Function under free disposal of outputs, one takes

$$x \in (L_\infty)^n_+ \to \mathbb{P}(x) = \{u \in (L_\infty)_+ : u \leqq \mathbb{F}(x)\} , \tag{2.5-6}$$

with $\mathbb{F}(x)$ given by (2.5-5), as a maximal output rate history .

When the dynamic neoclassical production function is looked at in this context it is easy to see that for a dynamic production correspondence with single output the function $\mathbb{F}(x)$ may not exist, since there may not be a total ordering of all output rate histories obtainable with $x \in (L_\infty)^n_+$ such that for any two output histories u and v , either $u(t) \geqq v(t)$ or $u(t) \leqq v(t)$ for all $t \in [0,+\infty)$. A simple example of this is given by the correspondence $x \in (L_\infty)_+ \to \mathbb{P}(x) \in 2^{(L_\infty)_+}$ where

$$\begin{aligned}\mathbb{P}(x) = \{w \in (L_\infty)_+ : w(t) = \theta u(t) , \theta \in [0,1] , t \in [0,+\infty)\} \cup \\ \{w \in (L_\infty)_+ : w(t) = \theta v(t) , \theta \in [0,1] , t \in [0,+\infty)\}\end{aligned} \tag{2.5-7}$$

and

$$\begin{aligned} u(t) &= \begin{cases} \alpha x(t) & \text{for } t \in [1,+\infty) , \alpha \in R_{++} \\ 0 & \text{for } t \in [0,1) \end{cases} \\ v(t) &= \begin{cases} \beta x(t) & \text{for } t \in [3,+\infty) , \beta > \alpha \\ 0 & \text{for } t \in [0,3) . \end{cases} \end{aligned} \tag{2.5-8}$$

It is rather direct to verify that the dynamic correspondence defined by (2.5-7) and (2.5-8) satisfies an acceptable combination of the axioms stated in Section 2.1-1. Clearly neither $\theta_1 u(t) \geqq \theta_2 v(t)$ or $\theta_1 u(t) \leqq \theta_2 v(t)$ for any pair $\theta_1 \in [0,1]$, $\theta_2 \in [0,1]$.

Here output is a single commodity, yet there is no consistent order relationship between output rate histories for the two possibilities u and v . This arises because of the differing time distributions for

the two output rate functions. In dynamic models "time substitution," i.e., substitution of one time distribution of output rate for another, is an important characteristic for outputs, and also inputs, as an additional dimension of economic alternatives to that provided by factor substitution where one may substitute the input (or output rate) history of one factor for that of another. Such factor substitutions may also involve time substitution in the factor exchanges made. Thus "time substitution" is truly an additional dimension of substitution for dynamic models of production over the simple factor substitution of steady state models.

The condition under which a dynamic version $x \varepsilon (L_\infty)^n_+ \to \mathbb{F}(x) \varepsilon (L_\infty)_+$ of the neoclassical production function may exist are given by the following proposition:

Proposition (2.5-1):

The dynamic neoclassical production function $\mathbb{F}(x)$ exists if and only if the parent dynamic correspondence $x \varepsilon (L_\infty)^n_+ \to \mathbb{P}(x) \varepsilon 2^{(L_\infty)_+}$ is such that $\mathbb{E}\mathrm{ff}\,\mathbb{P}(x) : = \{u \varepsilon \mathbb{P}(x) : v \notin \mathbb{P}(x) \text{ for } v \geq u\}$ consists of a single output rate function.

Proof:

Suppose $\mathbb{F}(x)$ exists. Then $u \leqq \mathbb{F}(x)$ for all $u \varepsilon \mathbb{P}(x)$, and $\mathbb{F}(x)$ represents a maximal output rate history. Accordingly if $v \varepsilon (L_\infty)_+$ and $v \geq u$, then $v \notin \mathbb{P}(x)$ and $\mathbb{F}(x)$ is a unique efficient output rate history. Conversely, if $\mathbb{E}\mathrm{ff}\,\mathbb{P}(x) = \{u\}$, $u \varepsilon (L_\infty)_+$, let $\mathbb{F}(x) : u$, and if $v \varepsilon \mathbb{P}(x)$, $v \leq \mathbb{F}(x)$, and $\mathbb{F}(x)$ is a maximal output rate history with $\mathbb{P}(x)$ expressed by (2.5-3) or (2.5-6).

It is concluded that the common practice of modifying the steady state production function (2.5-1) by (2.5-4), where output rate at any time t is made to depend upon input rates at the same time, and possibly t as well, is a specialization of the possibilities for a dynamic neoclassical production function. Moreover the latter, and perforce the specialization, can exist only under somewhat strongly restrictive circumstances, namely where the time distribution of output is such that there is an output-rate-dominating history among the various possible time distributions of output rate histories which may be obtained from the vector x of input rate histories. In Chapter 10 dealing with activity analysis models we shall see that this restriction is indeed a strong circumscription.

A related issue arises in connection with Technological Progress expressed by a macroeconomic neoclassical dynamic production function $\mathbb{F}(K,L) := F(K(t),L(t),t)$ as in

$$F(K(t),L(t),t) := A(t)F_1(K(t),L(t)) \quad \text{OUTPUT AUGMENTING}$$

$$F(K(t),L(t),t) := F_2(K(t),B(t)L(t)) \quad \text{LABOR AUGMENTING}$$

$$F(K(t),L(t),t) := F_3(C(t),K(t),L(t)) \quad \text{CAPITAL AUGMENTING.}$$

If the restrictive conditions for the existence of the production function $\mathbb{F}(x)$ are not satisfied, observed changes as reflected by these functions fitted to data may very well be the result merely of time substitution of output rate, instead of reflecting technological progress, or at least the latter may be confounded by the former.

The properties of a dynamic neoclassical production function $\mathbb{F}(x)$ implied by those of the correspondence $x \varepsilon (L_\infty)_+^n \to \mathbb{P}(x) \varepsilon 2^{(L_\infty)_+}$, $\mathbb{P}(x) = \{u \varepsilon (L_\infty)_+ : u = \theta\, \mathbb{F}(x)\ ,\ \theta \varepsilon [0,1]\}$ (or $\mathbb{P}(x) = \{u \varepsilon (L_\infty)_+ : u \leqq \mathbb{F}(x)\}$) are:

$\mathbb{F}.1$ $\mathbb{F}(0) = 0 \varepsilon (L_\infty)_+$

$\mathbb{F}.2$ $||F(x)|| < +\infty$ for $||x|| < +\infty$

$\mathbb{F}.3$ $\mathbb{F}(\lambda x) \geqq \mathbb{F}(x)$ for $\lambda \varepsilon [1,+\infty)$

$\mathbb{F}.4$ If $\mathbb{F}(\bar{\lambda} x) \geq 0$ for $\bar{\lambda} \varepsilon (0,+\infty)$, there exists for each $\theta \varepsilon (0,+\infty)$ a scalar λ_θ such that $\mathbb{F}(\lambda_\theta x) \geqq \theta\, \mathbb{F}(x)$.

$\mathbb{F}.5$ $\mathbb{F}(x)$ is upper semi-continuous on $(L_\infty)_+^n$, i.e., if $\{x^\alpha\} \to x^o$ then for each $t \varepsilon [0,+\infty)$ $\lim_{\alpha\to\infty} \text{Sup}\ F(x^\alpha,t) \leqq F(x^o,t)$ where $F(x^\alpha,t)$ is the value of $\mathbb{F}(x^\alpha)$ at time $t \varepsilon [0,+\infty)$.

Property $\mathbb{F}.1$ holds since $\mathbb{P}(0) = \{0\}$ by property $\mathbb{P}.1$ implies by either (2.5-3) (weak disposal of outputs) or (2.5-6) (strong disposal of outputs) that $\mathbb{F}(0) = 0 \varepsilon (L_\infty)_+$.

In the case of property $\mathbb{F}.2$, the property $\mathbb{P}.2$ for $x \to \mathbb{P}(x)$, namely $\mathbb{P}(x)$ is bounded for $||x|| < +\infty$, implies either by (2.5-6) or (2.5-3) that $||\,\mathbb{F}(x)|| < +\infty$.

In regard to property $\mathbb{F}.3$, one may invoke only the weak disposal of inputs property $\mathbb{P}.3$ for $x \to \mathbb{P}(x)$, i.e., $\mathbb{P}(\lambda x) \supset \mathbb{P}(x)$ for $\lambda \varepsilon [1,+\infty)$, to obtain from either (2.5-3) and (2.5-6) that $\mathbb{F}(\lambda x) \geqq \mathbb{F}(x)$ for $\lambda \varepsilon [1,+\infty)$.

Stronger properties $\mathbb{F}.3S$ and $\mathbb{F}.3SS$ for $\mathbb{F}(x)$ may be stated

$\mathbb{F}.3S \quad \mathbb{F}(\lambda_1 x_1, \ldots, \lambda_n x_n) \geqq \mathbb{F}(x)$ for $\lambda_i \in [1,+\infty)$, $i \in \{1, \ldots, n\}$

$\mathbb{F}.3SS \quad \mathbb{F}(x') \geqq \mathbb{F}(x)$ for $x' \geqq x$

by invoking properties $\mathbb{P}.3S$ and $\mathbb{P}.3SS$ for the parent correspondences (2.5-3) or (2.5-6).

Concerning property $\mathbb{F}.4$, property $\mathbb{P}.4.2$ for the parent correspondence $x \to \mathbb{P}(x)$ implies that if $\mathbb{P}(\bar{\lambda}x) \neq \{0\}$ for some $x \geq 0$ and $\bar{\lambda} > 0$, and $u^o \in \mathbb{P}(\bar{\lambda}x)$, then for all $\theta \in (0,+\infty)$ there exists a scalar λ_θ such that $(\theta u) \in \mathbb{P}(\lambda_\theta x)$. Accordingly for either (2.5-3) or (2.5-6), $\mathbb{F}(\lambda_\theta x) \geqq \theta u$, $||u|| > 0$, for $\theta \to +\infty$.

The upper semi-continuity of $\mathbb{F}(x)$ follows from property $\mathbb{P}.5$ (closure) of the parent correspondence $x \to \mathbb{P}(x)$. In order to verify this fact, consider the inverse correspondence defined by

$$u \in (L_\infty)_+ \to \mathbb{L}(u) = \left\{x \in (L_\infty)_+^n : \mathbb{F}(x) \geqq u\right\} \quad \text{or}$$

$$u \in (L_\infty)_+ \to \mathbb{L}(u) = \left\{x \in (L_\infty)_+^n : \lambda\, \mathbb{F}(x) = u \text{ , } \lambda \in [1,+\infty)\right\} .$$

The closure of $x \to \mathbb{P}(x)$ implies that $\mathbb{L}(u)$ is closed for $u \in (L_\infty)_+$. Suppose that $\mathbb{F}(x)$ is not upper semi-continuous at $x^o \in (L_\infty)_+^n$. Then for some $t^o \in [0,+\infty)$ there exists an infinite sequence $\{x^\alpha\} \to x^o$ such that $\lim\limits_{\alpha\to\infty} \text{Sup } F(x^\alpha,t^o) > F(x^o,t^o)$. Then for some value $\bar{u}(t^o) \in R_{++}$ there is a subsequence $\left\{x^{\alpha_k}\right\} \to x^o$ such that

$$\lim_{k\to\infty} F\left(x^{\alpha_k},t^o\right) = \lim_{\alpha\to\infty} \text{Sup } F(x^\alpha,t^o) > \bar{u}(t^o) > F(x^o,t^o) .$$

Then for $v \in (L_\infty)_+$ defined by

$$v(t) = \begin{cases} \bar{u}(t^o) & \text{for} \quad t = t^o \\ \lim\limits_{k\to\infty} F\left(x^{\alpha_k}, t\right) & \text{for} \quad t \neq t^o \end{cases}$$

there exists a positive integer K such that for $k \geqq K$, $\mathbb{F}\left(x^{\alpha_k}\right) \geqq v$, implying $x^{\alpha_k} \in \mathbb{L}(v)$ for all $k \geqq K$. But $x^o \notin \mathbb{L}(v)$, since $F(x^o, t^o) < \bar{u}(t^o)$, contradicting the closure of $\mathbb{L}(v)$ for $v \in (L_\infty)_+$. Thus the dynamic production function $\mathbb{F}(x)$ is upper semi-continuous on $x \in (L_\infty)^n_+$.

Additional properties for $\mathbb{F}(x)$ follow from the axioms for the correspondence (2.5-4) or (2.5-6):

$\mathbb{F}$.T.1 $F(x,t) = 0$ for $t \in [0, \bar{t}_u)$, $\bar{t}_u \in R_{++}$

$\mathbb{F}$.T.2 $F(x,t) = 0$ for $t > \bar{t}_x$

$\mathbb{F}$.T.3 If $u \in (L_\infty)_+$ is summable, there exists $x \in (L_\infty)^n_+$ such that x is summable and $\mathbb{F}(x) \geqq u$

$\mathbb{F}$.T.4 If $\mathbb{F}(x) \geqq u$, then $\mathbb{F}(y) \geqq u$ where $y_i(t) = x_i(t)$ for $t \in [0, \bar{t}_u)$, $y_i(t) = 0$ for $t \in [\bar{t}_u, +\infty)$, $i \in \{1, 2, \ldots, n\}$.

Here $F(x,t)$ is the value of $\mathbb{F}(x)$ at t.

2.6 Steady State (Static) Production Correspondence

In economic theory and econometric studies the production models (functions) used are quite often of the timeless variety, i.e., static or steady state representations. Therefore, in the context of the foregoing dynamic structure of production, it is of some interest to inquire into the existence of these static relationships and to consider the

possible definition of such production functions in terms of the fundamental underlying dynamic structure.

As before, let $u \in (L_\infty)^m_+ \to \mathbb{L}(u) \in 2^{(L_\infty)^n_+}$, $x \in (L_\infty)^n_+ \to \mathbb{P}(x) \in 2^{(L_\infty)^m_+}$, be a pair of inversely related dynamic production correspondences. In the case of a steady state model of this process, one is interested in vectors $v \in (L_\infty)^m_+$ of constant output rate functions. Because of the axiom $\mathbb{P}$.T.1 (see Section 2.1.3) the components of such functions must be zero for an initial interval. This peculiarity arises because the dynamic structure proceeds from a finite origin in time, which is not of any significance for the timeless models. Thus we consider the subset $C \subset (L_\infty)^m_+$ defined by

(2.6-1)
$$C := \left\{ v \in (L_\infty)^m_+ : v_i(t) = 0 \ , \ t \in \left[0, t_i^o\right) , v_i(t) = v_i^o \in R_+ \ , \right.$$
$$\left. t \in \left[t_i^o, +\infty\right) \ , \ i \in \{1,2, \ldots, m\} \right\} .$$

Clearly C is a closed positive subcone of $(L_\infty)^m_+$. On the input side constant rate input functions are of interest, with no restriction on initiation of positive values. Thus, we consider the closed positive subcone $C' \subset (L_\infty)^n_+$ defined by

(2.6-2) $$C' := \left\{ y \in (L_\infty)^n_+ : y_i(t) = y_i^o \in R_+ \ , \ t \in [0,+\infty) \ , i \in \{1,2, \ldots, m\} \right\} .$$

With this notation a steady state (static) production correspondence $u \in R^m_+ \to L(u) \in 2^{R^n_+}$ is defined by:

(2.6-3) For $v \in C$, $u := (||v_1||, ||v_2||, \ldots, ||v_m||)$.

$$(2.6\text{-}4)\quad L(u) := \left\{x \in R_+^n : x = (||y_1||, ||y_2||, \ldots, ||y_n||)\,,\, y \in \mathbb{L}(v) \cap C'\right\}.$$

This definition is not vacuous if $\mathbb{L}(u) \cap C'$ is not empty. Whether or not this may happen depends upon the axioms taken for the parent dynamic production correspondence. If free disposability of inputs is not assumed, i.e., only either $\mathbb{L}.3$ or $\mathbb{L}.3S$ are taken to apply, it is not denied that $\mathbb{L}(v) \cap C'$ is empty for all $v \in (L_\infty)_+^m$. On the other hand, if inputs are freely disposable, i.e., property $\mathbb{L}.3SS$ applies, then $\mathbb{L}(v) \cap C'$ is not empty for $\mathbb{L}(v)$ not empty, because if $y \in \mathbb{L}(v)$, the vector $z = (z_1, z_2, \ldots, z_n)$ defined by

$$z_i(t) := ||y_i||\,,\ t \in [0,+\infty)\,,\ i \in \{1,2, \ldots, n\}$$

likewise belongs to $\mathbb{L}(v)$, since $z \geqq y$. In such cases a feasible vector of constant input rates is taken as a vector of constant suprema input rates of a given vector of input functions. By this construction the alternative input vectors x *of* R_+^n *to yield the vector* $u \in R_+^m$ are taken as constant input rates to cover the maximal input rate for each input, in the case of each dynamic vector y of input rate functions yielding $v \in C$.

Efficiency of input vector, dynamic and static need not be preserved by this construction. It can happen that $\mathrm{Eff}\,\mathbb{L}(v) \cap C'$ is empty. Nothing in the axiomatic structure of the parent dynamic correspondence prevents this. However, if $\mathbb{L}(v) \cap C'$ is not empty in case only $\mathbb{L}.3$ and $\mathbb{L}.3S$ apply, or $\mathbb{L}.3SS$ applies, $L(u)$ corresponding to $v \in C$ is not empty and for $L(u)$ there exists an efficient vector x, i.e., $\mathrm{Eff}\ L(u)$ is not empty. Thus an efficient input vector x of $L(u)$

need not correspond to an efficient vector y of $\mathbb{L}(v)$ from which $L(u)$ is derived. However, if $y \in \text{Eff } \mathbb{L}(v) \cap C'$, then $x = (||y_1||, ||y_2||, \ldots, ||y_n||)$ clearly belongs to $\text{Eff } L(u)$. Although the possibility of the latter is not clear at the level of abstraction of this discussion, it will be seen later when Activity Analysis is considered (Chapter 10) that Time Substitution plays a role in this respect.

Certain questions still remain. Will the correspondence defined by (2.6-3), (2.6-4) satisfy the axioms for a steady state production correspondence $u \in R^m_+ \rightarrow L(u) \in 2^{R^n_+}$? [See (Shephard, 1974:a).] Will the output correspondence $x \in R^n_+ \rightarrow P(x) \in 2^{R^m_+}$ similarly defined be inverse to $u \rightarrow L(u)$?

For convenient reference, the properties required of $u \rightarrow L(u)$ are:

L.1 $L(0) = R^n_+$, $0 \notin L(u)$ for $u \geq 0$

L.2 $\bigcap_{\alpha=1}^{\infty} L(u^\alpha)$ is empty for $\{||u^\alpha||\} \rightarrow +\infty$

L.3 If $x \in L(u)$, $(\lambda x) \in L(u)$ for $\lambda \in [1,+\infty)$

L.3S If $x \in L(u)$ and $x' \geqq x$, then $x' \in L(u)$

L.4.1 For each $i \in \{1,2, \ldots, m\}$ there exists an input vector $x^{(i)} \in R^n_+$ such that $x^{(i)} \in L(u)$ for $u_i > 0$

L.4.2 If $(\lambda x) \in L(\bar{u})$ for $\lambda \in (0,+\infty)$, $\bar{u} \geq 0$, then $\{\lambda x \mid \lambda \in [0,+\infty)\} \cap L(\theta\bar{u})$ is not empty for $\theta \in [0,+\infty)$

L.5 If $\{u^\alpha\} \rightarrow u^o$, $x^\alpha \in L(u^\alpha)$ for $\alpha = 1,2, \ldots$, and $\{x^\alpha\} \rightarrow x^o$, then $x^o \in L(u^o)$

L.6 $L(\theta u) \subset L(u)$ for $\theta \in [1,+\infty)$

L.6S $L(u') \subset L(u)$ for $u' \geqq u$.

These properties are steady state versions of those listed in Section 2.2.1 for the dynamic production correspondence $u \varepsilon (L_\infty)_+^m \to \mathbb{L}(u) \varepsilon 2^{(L_\infty)_+^n}$, and conform to those in (Shephard, 1974:a).

In regards to property L.1, if $v = 0 \varepsilon C$, $\mathbb{L}(v) = (L_\infty)_+^m$ (see property $\mathbb{L}$.1, Section 2.2.1). Also $0 \notin \mathbb{L}(v)$ for $||v|| > 0$. Consequently,

$$L(0) = \left\{x \varepsilon R_+^n : x = (||y_1||,||y_2||, \ldots, ||y_n||) , y \varepsilon (L_\infty)_+^n \cap C' = C'\right\}$$
$$= R_+^n .$$

Further, since $0 \notin \mathbb{L}(v)$ for $||v|| > 0$ and $v \varepsilon C$, $0 \notin \mathbb{L}(v) \cap C'$ and $0 \notin L(u)$ for $u \geq 0$.

For consideration of property L.2, let $\{u^\alpha\} \subset R_+^m$ be an infinite sequence with $||u^\alpha|| \to +\infty$. Corresponding to u^α, consider $v^\alpha \varepsilon C$ where

$$||v_i^\alpha|| = u_i^\alpha .$$

Then $||v^\alpha|| \to +\infty$ and by property $\mathbb{L}$.2 (Section 2.2.1) $\bigcap_{\alpha=1}^{\infty} \mathbb{L}(v^\alpha)$ is empty. Now, suppose the contrary of L.2. Then there exists $\bar{x} \varepsilon L(u^\alpha)$ for all α. Accordingly consider $\bar{y} \varepsilon (L_\infty)_+^n$, defined by

$$\bar{y}_i(t) = \bar{x}_i , t \varepsilon [0,+\infty) , i \varepsilon \{1,2, \ldots, n\} .$$

Since $\bar{x} \varepsilon L(u^\alpha)$ for all α, $\bar{y} \varepsilon \mathbb{L}(v^\alpha)$ for all α, contradicting property $\mathbb{L}$.2. Thus property L.2 holds. The case where $\mathbb{L}(v) \cap C'$

is empty for $v \in C$ (properties $\mathbb{L}.3$ and $\mathbb{L}.3S$ only) is of no concern, since then L.2 is trivially satisfied.

Concerning property L.3, suppose $x \in L(u)$. Then $x = (||y_1||, ||y_2||, \ldots, ||y_n||)$ where $u = (||v_1||, \ldots, ||v_n||)$ with $v_i(t) = u_i$, $t \in [0,+\infty)$, $i \in \{1,2, \ldots, m\}$, and $y \in \mathbb{L}(v) \cap C'$. By property $\mathbb{L}.3$ (Section 2.2.1) for the parent dynamic correspondence, $\lambda y \in \mathbb{L}(v) \cap C'$ for $\lambda \in [1,+\infty)$, since $\lambda y \in \mathbb{L}(v)$ and $\lambda y \in C'$ when $\lambda \in [1,+\infty)$. Thus $z = (||\lambda y_1||, ||\lambda y_2||, \ldots, ||\lambda y_n||) = \lambda x$ belongs to $L(u)$ for $\lambda \in [1,+\infty)$. Hence property L.3 holds when $\mathbb{L}.3$ applies. Since $\mathbb{L}.3SS \Rightarrow \mathbb{L}.3S \Rightarrow \mathbb{L}.3$, the property L.3 for the steady state correspondence holds when either of these two stronger axioms for the dynamic correspondence apply.

In the case of property L.3S, one need only observe that $y \in \mathbb{L}(v) \cap C'$ implies $(\lambda_1 y_1, \ldots, \lambda_n y_n) \in \mathbb{L}(v)$, $(\lambda_1 y_1, \ldots, \lambda_n y_n) \in C'$ for $\lambda_i \in [1,+\infty)$, $i \in \{1,2, \ldots, n\}$, whence $x' = (\lambda_1 ||y_1||, \ldots, \lambda_n ||y_n||) \in L(u)$. Moreover all x' at least as large as x may be represented by $(\lambda_1 x_1, \ldots, \lambda_n x_n)$ for $\lambda_i \in [1,+\infty)$, $i \in \{1,2, \ldots, n\}$. Since $\mathbb{L}.3SS \Rightarrow \mathbb{L}.3S$, property L.3S holds for $u \to L(u)$ if properties $\mathbb{L}.3S$ or $\mathbb{L}.3SS$ apply for the parent dynamic correspondence.

Now property L.4.1 is not implied by the axiom $\mathbb{L}.4.1$ for the dynamic correspondence, $v \to \mathbb{L}(v)$, $v \in C$, since the latter assures only that $||v_i||$ can be made positive for $v_i \in (L_\infty)_+$, $i \in \{1,2, \ldots, m\}$. In order to have property L.4.1 hold, the axiom $\mathbb{L}.4.1$ must be strengthened to:

$$(\mathbb{L}.4.1S) \quad \Big\{x \in (L_\infty)_+^n : v_i(t) = 0\ ,\ t \in \left[0, t_i^o\right),\ v_i(t) = v_i^o \in R_{++}\ ,\ t \in \left[t_i^o, +\infty\right) \text{ for some } v \in \mathbb{P}(x)\Big\} \neq \emptyset\ ,\ i \in \{1,2, \ldots, m\} .$$

With this guarantee, each output may be produced with constant positive output rate when inputs are freely disposable for the parent dynamic production correspondence. However under the properties $\mathbb{L}.3$ and $\mathbb{L}.3S$, $\mathbb{L}(v) \cap C'$ may be empty, preventing a positive output rate in the steady state model, even when $\mathbb{L}.4.1S$ applies. Under the weaker axioms $\mathbb{L}.3$, $\mathbb{L}.3S$ for the dynamic correspondence one needs the stronger axiom

$$(\mathbb{L}.4.1SS) \quad \left\{x \in C' : v_i(t) = 0 \ , \ t \in \left[0, t_i^o\right), v_i(t) = v_i^o \in R_{++} \ , \ t \in \left[t_i^o, +\infty\right) \text{ for some } v \in \mathbb{P}(x)\right\} \neq \emptyset \ , \ i \in \{1,2, \ldots, m\}$$

for the dynamic correspondence in order to assure that L.4.1 holds for the derived steady state correspondence.

Next, concerning property L.4.2, let $x \in L(\bar{u})$ for $\bar{u} \geq 0$. Then $x = (||y_1||,||y_2||, \ldots, ||y_n||)$ where $\bar{u} = (||v_1||,||v_2||, \ldots, ||v_m||)$, $v \in C$, and $y \in \mathbb{L}(v) \cap C'$. By property $\mathbb{L}.4.2$ for the dynamic correspondence $v \to \mathbb{L}(v)$, $\{\lambda y : \lambda \in [0,+\infty)\} \cap \mathbb{L}(\theta v)$ is not empty for $\theta \in [0,+\infty)$. Moreover, $\lambda y \in C'$ for $\lambda \in [0,+\infty)$. Thus $\{\lambda y : \lambda \in [0,+\infty)\} \cap \mathbb{L}(\theta v) \cap C'$ is not empty for $\theta \in [0,+\infty)$. Consequently for each $\theta \in [0,+\infty)$, there exists a scalar λ_θ such that $(||\lambda_\theta y_1||,||\lambda_\theta y_2||, \ldots, ||\lambda_\theta y_n||) = \lambda_\theta x \in L(u)$, and thus L.4.2 holds. Whether or not $\mathbb{L}.3SS$ holds is of no consequence for this proposition since the premise assumes $x \in L(u)$.

In regard to property L.5, consider $\{u^\alpha\} \to u^o$, $x^\alpha \in L(u^\alpha)$, $\alpha = 1,2, \ldots$ and $\{x^\alpha\} \to x^o$. Then there exists for each α, $v^\alpha \in C$ such that $u^\alpha = \left(||v_1^\alpha||,||v_2^\alpha||, \ldots, ||v_m^\alpha||\right)$, with $x^\alpha = \left(||y_1^\alpha||,||y_2^\alpha||, \ldots, ||y_n^\alpha||\right)$ where $y^\alpha \in \mathbb{L}(v^\alpha) \cap C'$. Since $\{u^\alpha\} \to u^o$ and $v^\alpha \in C$ for all α, $\{v^\alpha\} \to v^o \in C$ with $u^o = \left(||v_1^o||,||v_2^o||, \ldots, ||v_m^o||\right)$. Also, since

$y^\alpha \in C'$ for all α and $\{x^\alpha\} \to x^o$, $\{y^\alpha\} \to y^o \in C'$ with $x^o = \left(||y_1^o||, ||y_2^o||, \ldots, ||y_n^o||\right)$. Moreover, by property $\mathbb{L}.5$ for the dynamic correspondence $v \to \mathbb{L}(v)$ it follows that $y^o \in \mathbb{L}(v^o)$ implying $x^o \in L(u^o)$. Thus property L.5 holds. Again whether or not $\mathbb{L}$.3SS holds is of no consequence since the premise of property L.5 assumes $x^\alpha \in L(u^\alpha)$ for all $\alpha = 1,2, \ldots$.

Finally, considering properties L.6 and L.6S, one needs correspondingly to invoke only properties $\mathbb{L}$.6 and $\mathbb{L}$.6S or $\mathbb{L}$.6SS for the parent dynamic correspondence. For example, suppose $x \in L(\theta u)$. Then $x = (||y_1||, \ldots, ||y_n||)$, $\theta u = (||\theta v_1||, \ldots, ||\theta v_n||)$, $v \in C$, and $y \in \mathbb{L}(\theta v) \cap C'$. By property $\mathbb{L}$.6 for the parent dynamic correspondence, $\mathbb{L}(\theta v) \cap C' \subset \mathbb{L}(v) \cap C'$ for $\theta \in [1,+\infty)$. Hence $y \in \mathbb{L}(v) \cap C'$ and $x \in L(u)$. Thus $L(\theta u) \subset L(u)$ for $\theta \in [1,+\infty)$. Since $\mathbb{L}$.6SS $\Rightarrow$ $\mathbb{L}$.6S $\Rightarrow$ $\mathbb{L}$.6, property L.6 holds when any one of these three axioms holds for the parent dynamic correspondence. In the case of property L.6S, one need only invoke either $\mathbb{L}$.6S or $\mathbb{L}$.6SS and carry out an analogous argument.

Thus the following proposition holds:

<u>Proposition (2.6-1)</u>:

The steady state production correspondence derived from a dynamic production structure $v \in (L_\infty)_+^m \to \mathbb{L}(v) \in 2^{(L_\infty)_+^n}$ by (2.6-1) $\cdots$ (2.6-4) satisfies the properties L.1, ..., L.6S (excepting L.4.1). If the property $\mathbb{L}$.4.1 is strengthened to $\mathbb{L}$.4.1S and $\mathbb{L}$.3SS holds for the parent dynamic correspondence, property L.4.1 holds for the derived steady state correspondence.

Sub-Proposition (2.6-1):

If only properties $\mathbb{L}.3$ or $\mathbb{L}.3S$ apply for the parent dynamic correspondence, L.4.1 holds for the derived steady state correspondence if property $\mathbb{L}.4.1$ is strengthened to $\mathbb{L}.4.1SS$.

Turning now to the inverse correspondence of $u \in R^m_+ \to L(u) \in 2^{R^n_+}$ derived from $v \in (L_\infty)^m_+ \to \mathbb{L}(v) \in 2^{(L_\infty)^n_+}$, it is defined by

$$(2.6\text{-}5) \qquad P(x) = \left\{u \in R^m_+ : x \in L(u)\right\} , \ x \in R^n_+ .$$

Now, referring to (2.6-3), (2.6-4) this statement of $\mathbb{P}(x)$ can be written

$$(2.6\text{-}6) \qquad \text{For} \quad y \in C' , \ x := (||y_1||, ||y_2||, \ldots , ||y_n||)$$

$$(2.6\text{-}7) \quad P(x) = \left\{u \in R^m_+ : u = (||v_1||, ||v_2||, \ldots , ||v_m||) , v \in \mathbb{P}(y) \cap C\right\} ,$$

since $v \in C$ and $y \in \mathbb{L}(v)$ implies $v \in \mathbb{P}(y)$. Also the requirement that $y \in C'$ and $v \in C$ is included.

Statements (2.6-6) and (2.6-7) are a definition of a steady state output correspondence $x \in R^n_+ \to P(x) \in 2^{R^m_+}$ in terms of the dynamic production correspondence $y \in (L_\infty)^n_+ \to \mathbb{P}(y) \in 2^{(L_\infty)^m_+}$ inverse to $u \in (L_\infty)^m_+ \to \mathbb{L}(u) \in 2^{(L_\infty)^n_+}$, and this correspondence is clearly inverse to the steady state input correspondence defined by (2.6-3), (2.6-4). The properties of the correspondence $x \to P(x)$ are derivable from those of the dynamic correspondence $y \to \mathbb{P}(y)$, or inferable from the

expression (2.6-5), as:

P.1 $P(0) = \{0\}$

P.2 $P(x)$ is bounded for $||x|| < +\infty$

P.3 $P(\lambda x) \supset P(x)$ for $\lambda \in [1,+\infty)$

P.3S $P(x') \supset P(x)$ for $x' \geqq x$

P.4.1 $x \in R^n_+ : u_i > 0$ for some $u \in P(x)$ is not empty for $i \in \{1,2, \ldots, m\}$

P.4.2 If $\bar{u} \in \mathbb{P}(x)$, $x \geq 0$, $\bar{u} \geq 0$, there exists for each scalar $\theta \in (0,+\infty)$ a scalar λ_θ such that $(\theta\bar{u}) \in P(\lambda_\theta \cdot x)$

P.5 If $\{x^\alpha\} \to x^o$, $u^\alpha \in P(x^\alpha)$ for $\alpha = 1,2, \ldots$, and $\{u^\alpha\} \to u^o$, then $u^o \in P(x^o)$

P.6 If $u \in P(x)$, $(\theta u) \in P(x)$ for $\theta \in [0,1]$

P.6S If $u \in P(x)$, $v \in P(x)$ for $0 \leqq v \leqq u$.

They are steady state versions of the axioms stated in Section 2.2.1 for the dynamic production correspondence $x \in (L_\infty)^n_+ \to \mathbb{P}(x) \in 2^{(L_\infty)^m_+}$, and conform to those in (Shephard, 1974:a).

Thus a metaeconomic foundation is provided for the frequently used steady state (static) production functions used in economic analysis and econometric studies. Without free disposal of inputs (property $\mathbb{L}$.3SS) in the underlying dynamic process, one does not have at this level of abstraction any assurance that the steady state model is not vacuous. This is not an encouragement to assume free disposability, but rather a warning.

Another way of looking at a steady state model of production is to regard it as a statement of long run average relationships. This approach finds its expression most easily in terms of single output processes. Here we shall take the underlying dynamic process to be

$y \in (L_\infty)_+^n \to \mathbb{P}(y) \in 2^{(L_\infty)_+}$ where

$$\mathbb{P}(y) = \{u \in (L_\infty)_+ : u = \theta\, \mathbb{F}(y) \ , \ \theta \in [0,1]\}$$

or

$$\mathbb{P}(y) = \{u \in (L_\infty)_+ : u \leqq \mathbb{F}(y)\} \ ,$$

corresponding to weak or free disposal of outputs, and $\mathbb{F}(y)$ is a neo-classical dynamic production function

$$\mathbb{F} : y \in (L_\infty)_+^n \to \mathbb{F}(y) \in (L_\infty)_+$$

with the properties $\mathbb{F}.1, \ldots, \mathbb{F}.5$, stated in Section 2.5. As notation, let $F(x,t)$ be the value of the output history $\mathbb{F}(x)$ at the time t.

For a long-run-average steady state model one considers $y \in C'$, $x = (||y_1||, ||y_2||, \ldots, ||y_n||)$ and defines a static model $x \in R_+^n \to \phi(x) \in R_+$ by

$$\phi(x) = \lim_{n\to\infty} \frac{1}{T^n} \int_0^{T^n} F(y,t)\,d\mu(t) \tag{2.6-8}$$

where $\{T^n\}$ is an arbitrary monotone sequence with $T^n \to +\infty$ as $n \to +\infty$, and $d\mu$ is a suitable measure, say Lebesgue measure dt on $[0,+\infty)$,

or a counting measure if need be. The expression (2.6-8) is a meaningful definition of the long run average output rate $\phi(x)$ if it can be shown that the limit exists and is unique for all sequences $\{T^n\}$.

By property $\mathbb{F}.2$ for $y \to \mathbb{F}(y)$ (see Section 2.5), $||\mathbb{F}(y)|| < +\infty$ for $||y|| < +\infty$. Hence the function $F(y,t)$ is bounded uniformly in t , and

$$(2.6\text{-}9) \qquad 0 \leqq \frac{1}{T^n}\int_0^{T^n} F(y,t)dt \leqq ||\,\mathbb{F}(y)||$$

for all n . Accordingly the sequence $\{I_n\}$, $n = 1,2, \ldots$, where

$$(2.6\text{-}10) \qquad I_n := \frac{1}{T^n}\int_0^{T^n} F(y,t)dt$$

is a bounded set of real numbers, and there exists a subsequence $\{n_k\}$ with $T^{n_k} \to +\infty$ for $k \to +\infty$, such that $\left\{I_{n_k}\right\}$ converge to a limit $\bar{u}$. Now suppose there is another subsequence $\{n_\ell\}$ with $T^{n_\ell} \to +\infty$, such at $\left\{I_{n_\ell}\right\}$ converges to a limit $\bar{\bar{u}}$. For each member $n_{\ell i}$ of the sequence $\{n_\ell\}$, let n_{ki} be the smallest number of n_k such that $n_{ki} \geqq n_{\ell i}$. Then $T^{n_{ki}} \geqq T^{n_{\ell i}}$ and $\frac{1}{T^{n_{ki}}} \leqq \frac{1}{T^{n_{\ell i}}}$ with

$$\int_0^{T^{n_{ki}}} F(y,t)dt \geqq \int_0^{T^{n_{\ell i}}} F(y,t)dt \, .$$

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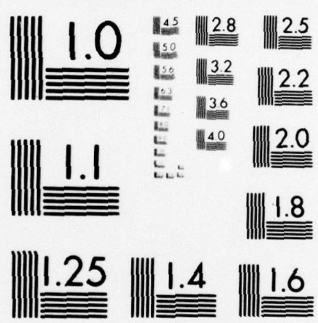

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Then for $(i = 1,2, \ldots)$

$$\frac{1}{T^{n_{ki}}}\int_0^{T^{n_{ki}}} F(y,t)dt - \frac{1}{T^{n_{\ell i}}}\int_0^{T^{n_{\ell i}}} F(y,t)dt \leqq$$

$$\frac{1}{T^{n_{\ell i}}}\int_0^{T^{n_{ki}}} F(y,t)dt - \frac{1}{T^{n_{\ell i}}}\int_0^{T^{n_{ki}}} F(y,t)dt = 0 \ .$$

Accordingly, $\bar{u} \leqq \bar{\bar{u}}$. By the same process, choosing for each item in $\{n_k\}$ the smallest element of $\{n_{\ell i}\}$ larger than the item in $\{n_k\}$, one may show $\bar{u} \geqq \bar{\bar{u}}$. Thus $\bar{u} = \bar{\bar{u}}$. Hence for the given monotone sequence $\{T^n\}$ the limit exists for the right hand side of (2.6-8). Now let $\{\bar{T}^n\}$ be any other monotone sequence with $\bar{T}^n \to \infty$ as $n \to \infty$. The two limits have to be equal, because the two sequences are merely subsequences of the monotone sequence $\{\bar{\bar{T}}^n\}$ which is the ordered union of $\{T^n\}$ and $\{\bar{T}^n\}$. Thus the long run average output rate $\phi(x)$ exists under rather general conditions for the underlying dynamic process.

It remains to consider the properties of the steady state (static) neoclassical production function $\phi(x)$ as a long run average output rate.

Let $\{T^n\}$ be an increasing sequence of real numbers with $T^n \to +\infty$. The properties of

$$\phi(x) := \lim_{n\to\infty} \frac{1}{T^n}\int_0^{T^n} F(y,t)dt$$

to be shown, are:

$\phi.1 \quad \phi(0) = 0$

$\phi.2 \quad \phi(x) < +\infty$ for $||x|| < +\infty$

$\phi.3 \quad \phi(\lambda x) \geqq \phi(x)$ for $\lambda \in [1,+\infty)$

$\phi.3S \quad \phi(\lambda_1 x_1, \ldots, \lambda_n x_n) \geqq \phi(x)$ for $\lambda_i \in [1,+\infty)$, $i \in \{1, \ldots, n\}$

$\phi.3SS \quad \phi(x') \geqq \phi(x)$ for $x' \geqq x$

$\phi.4 \quad$ If $\phi(x) > 0$, $\phi(\lambda x) \to +\infty$ as $\lambda \to +\infty$

$\phi.5 \quad \phi(x)$ is upper semi-continuous in x .

These properties conform to those stated ab initio for the static neoclassical production function. [See (Shephard, 1974:a).]

To show property $\phi.1$, note that by property $\mathbb{F}.1$ for the dynamic neoclassical production function that, $\mathbb{F}(0) = 0 \in (L_\infty)_+$. Now $x = 0$ implies $||y_i|| = 0$ for $i \in \{1,2, \ldots, n\}$ and $y = 0 \in (L_\infty)^n_+$. Thus $F(y,t) = F(0,t) = 0$ for almost all $t \in [0,+\infty)$ when $x = 0$. Accordingly $\phi(0) = 0$ by the definition (2.6-8).

In the case of property $\phi.2$, suppose $||x|| < +\infty$. This implies $||y|| < +\infty$ where $y_i(t) = x_i$, $i \in \{1, \ldots, n\}$ and by property $\mathbb{F}.2$ for the dynamic neoclassical production function, $||\mathbb{F}(y)|| < +\infty$. Hence $F(y,t)$ in (2.6-8) is bounded for all t in expression (2.6-8) and $\phi(x)$ is bounded. See (2.6-9).

For consideration of property $\phi.3$, let $\lambda \in [1,+\infty)$ and $\mathbb{F}(\lambda x) \geqq \mathbb{F}(x)$ by property $\mathbb{F}.3$ for the dynamic neoclassical production function. Hence, for $y_i(t) = x_i$, $i \in \{1,2, \ldots, n\}$,

$$\phi(\lambda x) = \lim_{n\to\infty} \frac{1}{T^n} \int_0^{T^n} F(\lambda y,t)dt \geqq \lim_{n\to\infty} \frac{1}{T^n} \int_0^{T^n} F(y,t)dt = \phi(x)$$

and $\phi.3$ holds. In the cases of $\phi.3S$ and $\phi.3SS$ one need only invoke the corresponding properties $\mathbb{F}.3S$ and $\mathbb{F}.3SS$ for the dynamic production function $\mathbb{F}(y)$. See Section 2.5.

Regarding property $\phi.4$, suppose $\phi(x) > 0$. Then clearly $\mathbb{F}(y) \neq 0$, where $y_i(t) = x_i$, $t \in [0,+\infty)$, $i \in \{1,2, \ldots, n\}$. Then, for all $\theta \in (0,+\infty)$ there exists a scalar λ_θ such that $\mathbb{F}(\lambda_\theta y) \geqq \theta\, \mathbb{F}(y)$. See property $\mathbb{F}.4$, Section 2.5. In particular consider a sequence $\{\theta^\nu\}$, $\nu = 1,2, \ldots$ with $\theta^\nu \to +\infty$. Corresponding to θ^ν there is a scalar λ^ν such that $\mathbb{F}(\lambda^\nu y) \geqq \theta^\nu\, \mathbb{F}(y)$ for $\nu = 1,2, \ldots$. Since $\mathbb{F}(y)$ is bounded and $\theta^\nu \to +\infty$, $\lambda^\nu \to +\infty$. Hence for almost all $t \in [0,+\infty)$ and $\{T^n\} \to +\infty$

$$\frac{1}{T^n}\int_0^{T^n} F(\lambda^\nu y,t)dt \geqq \theta^\nu \frac{1}{T^n}\int_0^{T^n} F(y,t)dt$$

and $\phi(\lambda^\nu x) \geqq \theta^\nu \phi(x)$. Thus $\phi(\lambda^\nu x) \to +\infty$ as $\lambda^\nu \to +\infty$.

Finally, concerning property $\phi.5$, let $\{x^\alpha\} \to x^o$ be an infinite sequence and define y^α by $y_i^\alpha(t) := x_i^\alpha \in R_+$, $t \in [0,+\infty)$ for $\alpha = 1,2, \ldots$ with y^o given by $y_i^o(t) := x_i^o$, $t \in [0,+\infty)$, $i \in \{1,2, \ldots, n\}$. Clearly $\{y^\alpha\} \to y^o$. By property $\mathbb{F}.5$ for the dynamic production function $\mathbb{F}(y)$,

$$\limsup_{\alpha\to\infty} F(y^\alpha,t) \leqq F(y^o,t) .$$

Hence

$$\lim_{\alpha\to\infty} \text{Sup}\ \phi(x^\alpha) = \lim_{\alpha\to\infty} \text{Sup}\left(\lim_{n\to\infty} \frac{1}{T^n}\int_0^{T^n} F(y^\alpha,t)dt\right)$$

$$\leqq \lim_{n\to\infty} \frac{1}{T^n}\int_0^{T^n} \lim_{\alpha\to\infty} \text{Sup}\ F(y^\alpha,t)dt$$

$$\leqq \lim_{n\to\infty} \frac{1}{T^n}\int_0^{\infty} F(y^o,t)dt = \phi(x^o)\ .$$

Thus property $\phi.5$ holds.

By the foregoing arguments one is then justified in using the traditional neoclassical static production function as a long run average in economic analysis, since the basic properties for this function follow as a consequence of those of the underlying dynamic structure.

In contrast one need not interpret the neoclassical production function $\phi(x)$ as a long run average. Using the construction of the first part of this section (see (2.6-6), (2.6-7)),

$$\text{For}\quad y \in C'\ ,\ x = (||y_1||,||y_2||,\ \ldots,||y_n||)$$

$$P(x) = \{u \in R_+ : u = ||v||\ ,\ v \in (\mathbb{P}(y) \cap C) \subset (L_\infty)_+\}$$

where in the case of weak disposal of outputs

$$\mathbb{P}(y) = \{v \in (L_\infty)_+ : v = \theta\,\mathbb{F}(y)\ ,\ \theta \in [0,1]\}$$

or in the case of free disposal of outputs

$$\mathbb{P}(y) = \{v \in (L_\infty)_+ : v \leqq \mathbb{F}(y)\}$$

and $\mathbb{F}(y)$ is the dynamic neoclassical production function. Now, when inputs are freely disposable, constant input rates are not prohibitive of yielding constant output rates. Hence $\mathbb{P}(y) \cap C$ is not ordinarily empty, and the static neoclassical production function is then given by

$$F(x) = ||\,\mathbb{F}(y)\,|| \tag{2.6-11}$$

where $y_i(t) = x_i$, $t \in [0,+\infty)$, $i \in \{1, \ldots, n\}$, is the maximal constant output rate obtainable with $y \in (L_\infty)^n_+$.

It is worth noting that although the production function (2.6-11) satisfies the properties ϕ.1, ..., ϕ.5 stated above, nothing is guaranteed strictly about the dynamic meaningfulness of the efficient subsets of

$$\begin{aligned} L(u) &= \left\{x \in R^n_+ : u \in P(x)\right\} \\ &= \left\{x \in R^n_+ : x = (||y_1||, \ldots, ||y_n||) \ , \ v \leqq \mathbb{F}(y)\right\} \ , \end{aligned}$$

where $v_i(t) = u_i$, $t \in [0,+\infty)$, $i \in \{1,2, \ldots, m\}$. As pointed out earlier, efficient points of $\mathbb{P}(y) \cap C$ need not correspond to efficient points of $L(u)$, and vice versa. In the case of the long run average definition of the production function, the notion of efficiency does not apply directly. For these reasons caution is needed in making determinations of efficient allocations (and optima related thereto) with the static neoclassical production function, if not downright avoidance, by either of the two foregoing approaches.

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